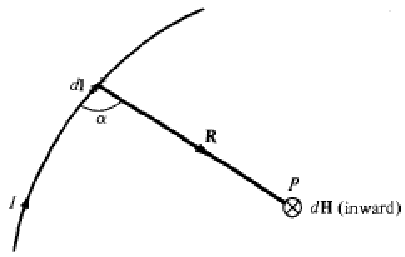
PHYS323

Electromagnetic Field Theory II

The Biot-Savart Law

The **Biot-Savart Law** is an equation that describes the magnetic field created by a current-carrying wire, and allows you to calculate its strength at various points. It looks like this

$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$



$$dH = \frac{kI\,dl \sin\alpha}{R^2}$$

where k is the constant of proportionality. In SI units, $k = 1/4\pi$,

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$

$$\boxed{d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}}$$

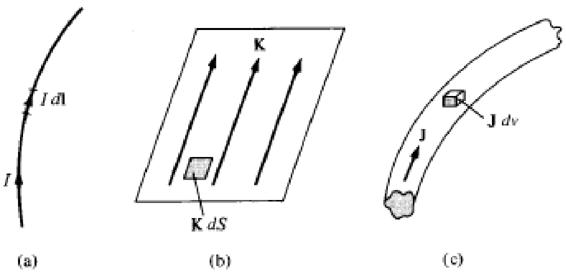
where $R = |\mathbf{R}|$ and $\mathbf{a}_R = \mathbf{R}/R$. Thus the direction of $d\mathbf{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $d\mathbf{H}$

in terms of the distributed current sources, the Biot–Savart law becomes

$$\mathbf{H} = \int_L \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(line current)}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\,dS \times \mathbf{a}_R}{4\pi R^2} \quad \text{(surface current)}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\,dv \times \mathbf{a}_R}{4\pi R^2} \quad \text{(volume current)}$$



Current distributions: **(a)** line current, **(b)** surface current, **(c)** volume current.

Conventional representation of **H** (or I) **(a)** out of the page and **(b)** into the page.

determine the field due to a straight current carrying filamentary conductor of finite length AB as in Figure

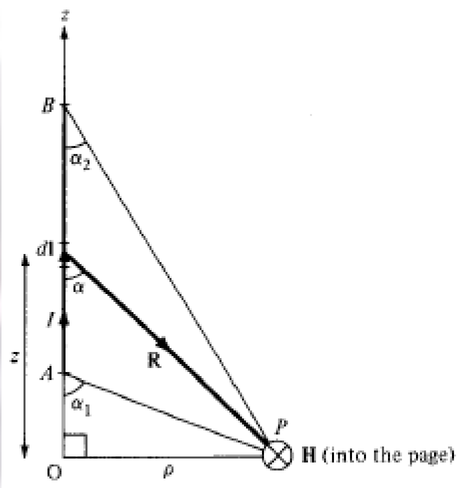
$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

But $d\mathbf{l} = dz\,\mathbf{a}_z$ and $\mathbf{R} = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, so

$$d\mathbf{l} \times \mathbf{R} = \rho\,dz\,\mathbf{a}_\phi$$

Hence,

$$\mathbf{H} = \int \frac{I\rho\,dz}{4\pi[\rho^2 + z^2]^{3/2}}\,\mathbf{a}_\phi$$



Letting $z = \rho\cot\alpha$, $dz = -\rho\,\mathrm{cosec}^2\,\alpha\,d\alpha$,

$$\mathbf{H} = -\frac{1}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\,\mathrm{cosec}^2\,\alpha\,d\alpha}{\rho^3\,\mathrm{cosec}^3\,\alpha}\,\mathbf{a}_\phi$$
$$= -\frac{I}{4\pi\rho}\,\mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha\,d\alpha$$

or

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi}$$

for any straight filamentary conductor of finite length

when the conductor is *semiinfinite* point *A* is now at point *A* is now at (90, 0, 0) while *B* is at (0, 0, ∞); α_1, = 90°, α_2 = 0°,

$$\mathbf{H} = \frac{I}{4\pi\rho}\,\mathbf{a}_\phi$$

when the conductor is *infinite* in length: *A* is at (0, 0, - ∞) while *B* is at (0, 0, , ∞); α_1= 180°, α_2 = 0°,

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi}$$

where a_l is a unit vector along the line current and a_ρ is a unit vector along the perpendicular line from the line current to the field point.

$$\boxed{\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho}$$

For a circular loop

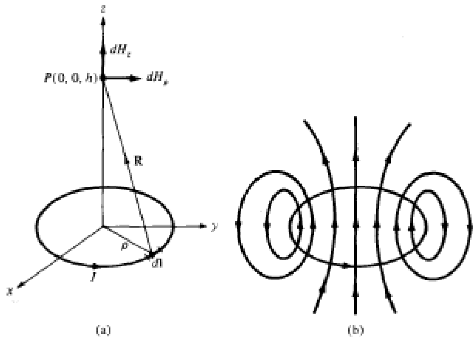
$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho \mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}} (\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$



(a) circular current loop, **(b)** flux lines due

By symmetry, the contributions along a_ρ add up to zero because the radial components produced by pairs of current element 180° apart cancel.

Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

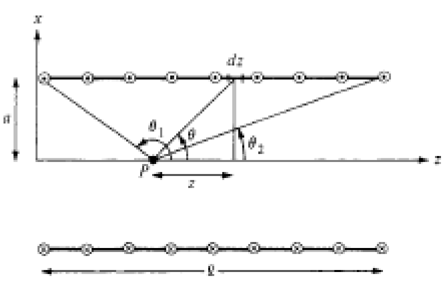
$$\mathbf{H} = \int dH_z\, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\, d\phi\, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

For solenoid of length l and radius *a* consists of N turns of wire carrying current

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$



If l>> a,

$$\mathbf{H} = nI\mathbf{a}_z$$

AMPERE'S CIRCUIT LAW

Ampere's circuit law states that the line integral of the tangential component of H around a *dosed* path is the same as the net current Ienc. enclosed by the path.

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}}$$

By applying Stoke's theorem

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

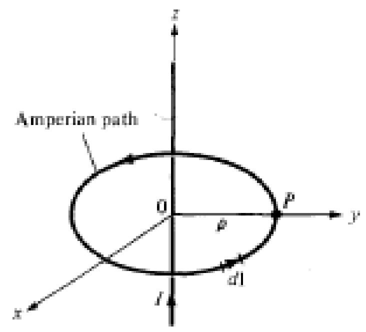
$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

APPLICATIONS OF AMPERE'S LAW

A. Infinite Line Current

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$



$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$

B. Infinite Sheet of Current

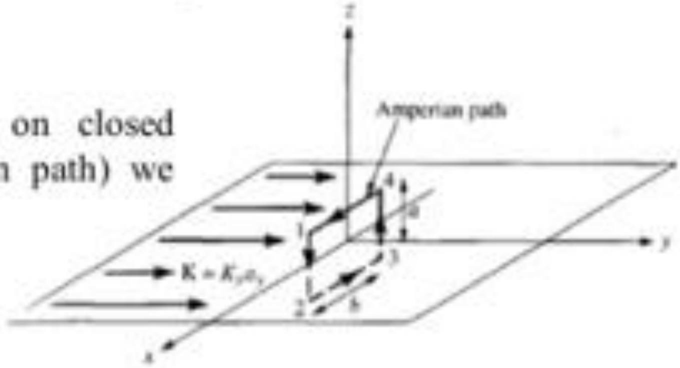
Consider an infinite current sheet in z = 0 plane.

If the sheet has a uniform current density then

$$\vec{K} = K_y \, \hat{a}_y$$

Applying Ampere's Law on closed rectangular path (Amperian path) we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b \quad \text{(i)}$$



To solve integral we need to know how **H** is like

We assume the sheet comprising of filaments $d\mathbf{H}$ above and below the sheet due to pair of filamentary current.

The resultant $d\mathbf{H}$ has only an x-component.

Also **H** on one side of sheet is the negative of the other.

Due to infinite extent of the sheet, it can be regarded as consisting of such filamentary pairs so that the characteristic of **H** for a pair are the same for the infinite current sheets

$$\mathbf{H} = \begin{cases} H_o\mathbf{a}_x & z > 0 \\ -H_o\mathbf{a}_x & z < 0 \end{cases} \qquad \text{(ii)}$$

where H_o is to be determined.

Evaluating the line integral of **H** along the closed path

$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l}$$

$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b)$$

$$= 2H_o b \qquad \text{(iii)}$$

Comparing (i) and (iii), we get

$$H_o = \frac{1}{2} K_y \qquad \text{(iv)}$$

Using (iv) in (ii), we get

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

C. Infinitely Long Coaxial Transmission Line

Generally, for an infinite sheet of current density **K**

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n$$

where a_n is a unit normal vector directed from the current sheet to the point of interest.

The magnetic flux density **B** is similar to the electric flux density **D** Therefore, the magnetic flux density **B** is related to the magnetic field intensity **H**

$$\mathbf{B} = \mu_o \mathbf{H}$$

where μ_o is a constant and is known as the permeability of free space. Its unit is Henry/meter (H/m) and has the value

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

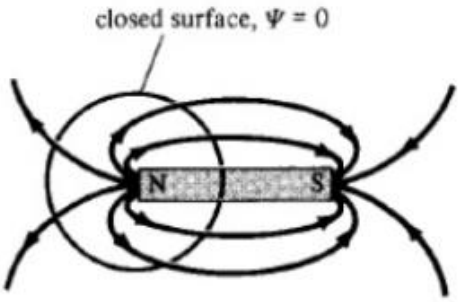
The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

where the magnetic flux ψ is in webers (Wb) and the magnetic flux density is in weber/ square meter or Teslas.

Magnetic flux lines are always close upon themselves,.

So it is not possible to have an isolated magnetic pole (or magnetic charges)



An isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero.

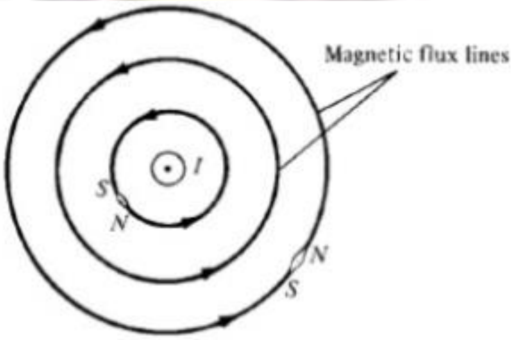
$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

This equation is known as the law of conservation of magnetic flux or Gauss's Law for Magnetostatic fields.

Magnetostatic field is not conservative but magnetic flux is conserved.

Magnetic flux lines due to a straight wire with current coming out of the page

Each magnetic flux line is closed with no beginning and no end and are also not crossing each other.

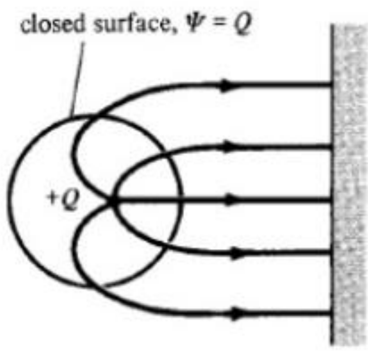


In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed.

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$$

Thus it is possible to have an isolated electric charge.

Also the electric flux lines are not necessarily closed.



Applying Divergence theorem, we get

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

or
$$\nabla \cdot \mathbf{B} = 0$$

This is Maxwell's fourth equation.

This equation suggests that magnetostatic fields have no source or sinks.
Also magnetic flux lines are always continuous.

According to Faraday a time varying magnetic field produces an induced voltage (called electromotive force or emf) in a closed circuit, which causes a flow of current.

The induced emf (V_{emf}) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is Faraday's Law and can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

where N is the number of turns in the circuit and ψ is the flux through each turn.

The negative sign shows that the induced voltage acts in such a way to oppose the flux producing in it. This is known as Lenz's Law.

For a circuit with a single turn (N = 1)

$$V_{\text{emf}} = -\frac{d\Psi}{dt}$$

In terms of **E** and **B** this can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(i)}$$

where ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by a closed path L..

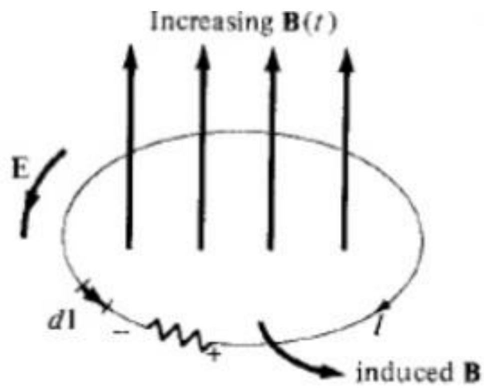
The equation says that in time-varying situation, both electric and magnetic fields are present and are interrelated.

The variation of flux with time may be caused in three ways.

1. By having a stationary loop in a time-varying **B** field.
2. By having a time-varying loop area in a static **B** field.
3. By having a time-varying loop area in a time-varying **B** field.

Consider a stationary conducting loop in a time-varying magnetic **B** field. The equation (i) becomes

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$



This emf induced by the time-varying current in a stationary loop is often referred to as transformer emf in power analysis since it is due to the transformer action.

By applying Stokes's theorem to the middle term, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is one of the Maxwell's equations for time-varying fields.

It shows that the time-varying field is not conservative.

$$\nabla \times \mathbf{E} \neq 0$$

When a conducting loop is moving in a static **B** field, an emf is introduced in the loop.

The force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

The motional electric field $\mathbf{E}_m$ is defined as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

Consider a conducting loop moving with uniform velocity **u,** the emf induced in the loop is

$$V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(i)}$$

This kind of emf is called the motional emf or flux-cutting emf. Because it is due to the motional action. eg,. Motors, generators

By applying Stokes's theorem to equation (i), we get

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})$$

Consider a moving conducting loop in a time-varying magnetic field

Then both transformer emf and motional emf are present.

Thus the total emf will be the sum of transformer emf and motional emf

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

also

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

For static EM fields

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \text{(i)}$$

But the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \qquad \text{(ii)}$$

But the continuity of current requires

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad \text{(iii)}$$

Equation (ii) and (iii) are incompatible for time-varying conditions

So we need to modify equation (i) to agree with (iii)

Add a term to equation (i) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \qquad \text{(iv)}$$

where $\mathbf{J}_d$ is to defined and determined.

Again the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \qquad \text{(v)}$$

In order for equation (v) to agree with (iii)

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$\text{or} \qquad \mathbf{J}_d = \frac{\partial \mathbf{D}}{dt} \qquad \text{(vi)}$$

Putting (vi) in (iv), we get

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

This is Maxwell's equation (based on Ampere Circuital Law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as displacement current density and $\mathbf{J}$ is the conduction current density $\mathbf{J} = \sigma \mathbf{E}$.

Maxwell's Equations in Final Form

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law