Approximation Theory

- References: المصادر

1- Partical Guide to Splines.
   By: Carl de Boor

2- Spline Analysic.
   By: M. H Schultz

3- Introdnction to Numerical Analysis.
   By: J. Stoer and R. Bulirsch

4- An Introdnction to Spline for use in Computer graphics and geometric modeling.
   By: Barsky

Syllbus: المنهج

1- General Idea of approximation theory and polynomail interpolation.

2- Spline Approximation الموضوع الرئيسي

3- Applications: Spline interpolation, numerical solution of ODE, PDE and integral equation.

General Idea of approximation theory and Polynomail interpolation

Use of approximation theory:

1- Replacement of difficult function. استبدال الدوال الصعبة.

2- Solution of ODE or PDE, I.E (Integral equation)

3- Compute certain function such that e^x, $\log x$, z^x, ...

Type of approximation Theory

1- Sucessive approximation or Iteration.
تقريب متتالي او متكرر

2- power series such as Taylar series.

3- Exponential or orthogoral functions.

4- Rational functions.

Taylar Series Exponsion :-

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^n(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)} f^{n+1}(\xi)$$

$$x_0 < \xi < x_1$$

Note :- - الدالة التي يكون لها نشر Taylar يجب أن تكون analytsic.

- ليس كل دالات لها نشر Taylar.

- ال Taylar في أساسه ال polynomal.

If we truncated at $(x-x_0)$, then we get a polynomail of degree n $P_n(x)$ with error $e_n(x)$

$$|e_n(x)| = |f(x) - P_n(x)| \leq \left|\frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi)\right|$$

error قانون ال

Disadvantages: السلبيات

1- n is often large for practical purposes.

2- Truncation error has Small radious of Convergence.

Example:

$\left(\frac{1+3x}{1+x}\right)^{\frac{1}{2}}$ well behaved in $(0, \infty)$

(T.E) Convergence only $(0, \frac{1}{3})$

Chapter -1-

"Polynomail Interpolation"

- one use polynomails for approximation because they can be evaluated, differenatiated anel integrate easily anel finitely many steps using the basic arithmetic operations of addition, Solutraction anel multiplication.

- A polynomail of order n in a function of the form

$$P(x) = a_1 + a_2 x + \cdots + a_n x^{j-1}$$

$$= \sum_{j=1}^{n} a_j x^{n-1}$$

is polynomail of degree $< n$.

Note: order $\cong$ No. of Cofficients.

$P_n = \{ p : P(x) = \sum_{j=1}^{n} a_j x^{j-1} \}$ set of all polynomail of order n anel degree $\leq n-1$.

Remark: Lagrange and Newton interpolation function are two representation (تمثيلين) of the same polynomail.

"Lagrange Interpolation"

Lagrange form Let $\tau = (\tau_i)_{i=1}^{n}$ be sequence of n distanct nodes then

$$l_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - \tau_j}{\tau_i - \tau_j}$$

product

is the Lagrange polynomail for τ.

- It is a polynomail of order n and vanishes at all τ_j is except for τ_i at which it takes the value 1.

- We write them with the aid of the kronecker delta as

$$l_i(\tau_i) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Hence for an arbitrary function g

$$p = \sum_{i=1}^{n} g(\tau_i)\, l_i(x) \quad \rightsquigarrow \text{Lagrange}$$

is an element of $\mathbb{P}_n$ and satisfies

$$p(\tau_i) = g(\tau_i) \qquad i = 1, 2, \dots, n$$

In the way we obtain for arbitrary g a function p in $\mathbb{P}_n$ that matches at n nodes $\tau_1, \tau_2, \dots, \tau_n$

x	$g(x)$
x_0	$g(x_0)$
x_1	$g(x_1)$
x_2	$g(x_2)$

$$p_2(x) = \sum_{i=0}^{2} g(x_i)\, l_i(x)$$

$$= g(x_0)\, l_0(x) + g(x_1) l_1(x) + g(x_2)\, l_2(x)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$p_2(x) = g(x_0)\,l_0(x) + g(x_1)\,l_1(x) + g(x_2)\,l_2(x)$$

$$p_2(x_0) = g(x_0)\,l_0(x) + g(x_0)\,l_1(x) + g(x_0)\,l_2(x)$$

$$= g(x_0)$$

$$p_2(x_1) = g(x_1), \quad p_2(x_2) = g(x_2)$$

Theorem: If $\tau_1, \tau_2, \ldots, \tau_n$ are distinct nodes and $g(\tau_1), g(\tau_2), \ldots, g(\tau_n)$ are the given data then there exists exactly on polynomail $p \in \mathbb{P}_n$ for

$$p(\tau_i) = g(\tau_i) \qquad i = 1, 2, \ldots, n$$

This polynomail can be written in Lagrange form

$$p = \sum_{i=1}^{n} g(\tau_i)\, l_i \quad \text{with} \quad l_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - \tau_j}{\tau_i - \tau_j}$$

Proof

Divided Difference and Newton form:

Def:- The kth divided difference of functions g at the nodes $\tau_i, \ldots, \tau_{i+k}$ is the leading coefficient (that is coefficient of x^k) of the polynomial of order $k+1$ that degree with g at the sequance $(\tau_i, \ldots, \tau_{i+k})$ It is denoted by $[\tau_i, \ldots, \tau_{i+k}]g$

This defention has the following immediate Can Sequance

i) If $p_i \in P_{<i}$ degree with g at $\tau_i, \ldots, \tau_{i+k}$ for $i=k$ and $k+1$ then:

$$p_{k+1}(x) = p_k(x) + (x-\tau_1)\cdots(x-\tau_k)[\tau_1, \ldots, \tau_{k+1}]g$$

Error

The Newton form can be given by

$$p_n(x) = \sum_{i=1}^{n} (x-\tau_1)\cdots(x-\tau_{i-1})[\tau_1, \ldots, \tau_i]g$$

first data $= [\tau_1]g + (x-\tau_1)[\tau_1, \tau_2]g +$

$$(x-\tau_1)(x-\tau_2)[\tau_1, \tau_2, \tau_3]g + \cdots +$$

$$(x-\tau_1)\cdots(x-\tau_{n-1})[\tau_1, \tau_2, \ldots, \tau_n]g$$

x	$g(x)$	
x_0	$g(x_0)$	$\rightarrow \frac{g(x_1)-g(x_0)}{x_1-x_0} = [x_0, x_1]g$
x_1	$g(x_1)$	$\rightarrow \frac{g(x_2)-g(x_1)}{x_2-x_1} = [x_1, x_2]g$
x_2	$g(x_2)$	$\rightarrow \frac{g(x_3)-g(x_2)}{x_3-x_2} = [x_2, x_3]g$
x_3	$g(x_3)$	

$$\frac{[x_1, x_2]g - [x_0, x_1]g}{x_2 - x_0} = [x_0, x_1, x_2]g$$

$$\frac{[x_2, x_3]g - [x_1, x_2]g}{x_3 - x_1} = [x_1, x_2, x_3]g$$

Properties of divided differences

1. If g is polynomial of degree $\leq k$ then $[\tau_i, \dots, \tau_{i+k}]$ is a constant as a function $\tau_i, \dots, \tau_{i+k}$
$[\tau_i, \dots, \tau_{i+k}]g = 0 \quad \forall g \in \mathbb{P}_{k-1}$

2. $[\tau_i, \dots, \tau_{i+k}]$ is a symmetric function of its arguments $\tau_i, \dots, \tau_{i+k}$ that is it depends only on the numbers $\tau_i, \dots, \tau_{i+k}$ and not on the order in which they occure in the argument list.

3. $[\tau_i, \dots, \tau_{i+k}]g$ is a linear in g, that is if $f = \alpha g + \beta h$ for some functions g and h and some number α and β then

$$[\tau_i, \dots, \tau_{i+k}]f = \alpha[\tau_i, \dots, \tau_{i+k}]g + \beta[\tau_i, \dots, \tau_{i+k}]h$$

4. Lebniz formula

If $f = gh$ that is $f(x) = g(x)h(x)$ for all x then

$$[\tau_i, \dots, \tau_{i+k}]f = \sum_{r=i}^{i+k} ([\tau_i, \dots, \tau_r]g)([\tau_r, \dots, \tau_{r+k}]h)$$

proof> observe that the function (x) agrees with f at $\tau_i, \dots, \tau_{i+k}$

$$\left[\sum_{i=1}^{i+k} (x-\tau_i)\cdots(x-\tau_{r-1})[\tau_i, \dots, \tau_r]g \cdot \sum_{s=i}^{i+k} (x-\tau_{s+1})\cdots(x-\tau_{i+k})[\tau_s, \dots, \tau_{i+h}]h\right] \dots * \text{ by Newton form}$$

$$\left[p_n(x) = \sum_{i=1}^{n} (x-\tau_i)\cdots(x-\tau_{i-1})[\tau_1, \dots, \tau_i]g\right]$$

the first factor agrees with g and the second factor agree with n

$$\sum_{r,s=i}^{i+k} = \sum_{r \le s} + \sum_{r > s}$$

Note that $\sum_{r>s}$ vanishes at $\tau_i, \dots, \tau_{i+k}$ also $\sum_{r \le s}$ must agree with f but $\sum_{r \le s}$ is a polynomail of order $k+1$

5- $[\tau_i, \ldots, \tau_{i+k}]$ is a Continuous function of it's $k+1$ agruments in case $g \in C^{(k)}$ that is g has k Continuous derivatives.

6- If $g \in C^{(k)}$, that is g has k Continuous derivatives then there exists a node ξ in the smallest interval Containing $\tau_i, \ldots, \tau_{i+k}$ so that $[\tau_i, \ldots, \tau_{i+k}]g = \frac{g^{(k)}(\xi)}{k!}$

7- For Computations it is importent to node that

$[\tau_i, \ldots, \tau_{i+k}]g = \frac{g^{(k)}(\tau_j)}{k!}$ if $\tau_i, \tau_{i+1} = \ldots = \tau_{i+k}$

في حال النقاط متكررة

$g \in C^{(k)}$ while

$$[\tau_i, \ldots, \tau_{i+k}]g = \frac{[\tau_i, \ldots, \tau_{r+1}, \tau_{r-1}, \ldots, \tau_{i+k}]g - [\tau_i, \ldots, \tau_{s-1}, \tau_{s+1}, \ldots, \tau_{i+k}]g}{\tau_r - \tau_s}$$

if τ_r, τ_s are any two distinct nodes in the sequance $\tau_i, \ldots, \tau_{i+k}$.

Theorem (osculatory interpolation)

If $g \in C^{(k)}$ that is g has n continuous derivative and $\tau = (\tau_i)_{i=1}^{n}$ is a sequance of n arbitrary nodes (not necessarily distinct) then for all x

$$g(x) = p_n(x) + (x-\tau_1)\cdots(x-\tau_n)[\tau_1,\ldots,\tau_n,x]g \quad \text{--- (1)} \quad \text{with}$$

$$p_m(x) = \sum_{i=1}^{m} (x-\tau_1)\cdots(x-\tau_{i-1})[\tau_1,\ldots,\tau_i]g \quad \text{--- (2)}$$

In particular p_n is the unige polynomail of order n that agree with g at τ.

Proof By mathematical induction it is true $n=1$

$$g(x) = g(\tau_1) + (x-\tau_1)[\tau,x]g \quad \text{--- (3)}$$

Suppose it is true for $n=k$

$$g(x) = p_k(x) + (x-\tau_1)\cdots(x-\tau_k)[\tau_1,\tau_2,\ldots,\tau_k,x]g \quad \text{--- (4)}$$

then apply (3) with (τ_1 replaced by τ_{k+1}) to $[\tau_1,\tau_2,\ldots,\tau_k,x]g$ as a function of x we get

Let $f(x) = [\tau_1,\ldots,\tau_k,x]g$ with $\tau_1 = \tau_{k+1}$ in (3) then

$$f(x) = f(\tau_{k+1}) + (x-\tau_{k+1})[\tau_{k+1},x]f \quad \text{therefore}$$

$$[\tau_1,\ldots,\tau_k,x]g = [\tau_1,\ldots,\tau_{k+1},x]g + (x-\tau_{k+1})[\tau_{k+1},x][\tau_1,\ldots,\tau_k,x]g \quad \text{--- (5)}$$

Sub (5) in (4)

$$g(x) = p_k(x) + (x-\tau_1),\ldots,(x-\tau_k)\Big[[\tau_1,\tau_2,\ldots,\tau_n]g + (x-\tau_{k+1})[\tau_{k+1},x][\tau_1,\ldots,\tau_k,x]g\Big]$$

$$= p_k(x) + (x-\tau_1),\ldots,(x-\tau_k)[\tau_1,\tau_2,\ldots,\tau_{k+1}]g + (x-\tau_1)(x-\tau_2)\cdots(x-\tau_{k+1})[\tau_{k+1},x][\tau_1,\ldots,\tau_k,x]g$$

$$g(x) = P_{k+1} + (x-\tau_1),\ldots,(x-\tau_{k+1})[\tau_{k+1},x][\tau_1,\ldots,\tau_k,x]g$$

the polynomail

$$P_{k-1}(x) + (x-\tau_1)\cdots(x-\tau_{k+1})[\tau_{k+1},y][\tau_1,\ldots,\tau_k,\cdot]g$$

is a polynomail of degree $k+1$ and order $k+2$ agree with g at $\tau_1,\ldots,\tau_{k+1}$ and at y.

$\blacksquare$

Divided difference table, the Cofficient

$[\tau_1]g$, $[\tau_1,\tau_2]g, \dots, [\tau_1,\tau_2,\dots,\tau_n]g$ for the Newton form are efficiently computed in a divided difference table : interpolation

nodes	value	1st d.d	2nd d.d	$(n-2)^{nd}$ d.d
τ_1	$g(\tau_1)$	$[\tau_1,\tau_2]g$	$[\tau_1,\tau_2,\tau_3]g$	$[\tau_1,\dots,\tau_{n-1}]g$
τ_2	$g(\tau_2)$	$[\tau_2,\tau_3]g$	$[\tau_{n-2},\tau_{n-1},\tau_n]g$	
$\vdots$	$\vdots$			
τ_{n-1}	$g(\tau_{n-1})$	$[\tau_{n-1},\tau_n]g$	$[\tau_2,\dots\tau_n]g$	
τ_n	$g(\tau_n)$			

$(n-1)^{st}$ d.d

$$[\tau_1,\dots,\tau_n]g$$

Then on Computing $[\tau_i,\tau_{i+1},\dots,\tau_{i+r}]$ either $\tau_i=\tau_{i+1}=\dots=\tau_{i+r}$ in which case

$$[\tau_i,\dots,\tau_{i+r}]g = \frac{g^{(r)}[\tau_i]}{r!}$$ or else (distinct)

$$[\tau_i,\dots,\tau_{i+r}]g = \frac{[\tau_{i+1},\dots,\tau_{i+r}]g - [\tau_i,\dots,\tau_{i+r+1}]g}{\tau_{i+r}-\tau_i}$$

Example: (Osulatory interpolation to Logarithm)

For $g(x) = \ln(x)$ estimate $g(1.5)$ by the number $P_4(1.5)$ with P_4 in the Cubic polynomial that agree with g at 1, 1, 2, 2 we set up the divided difference table

T	$g(T)$	1st d.d	2nd d.d	3rd d.d
1	0			
		→ 1		
1	0		→ −0.306853	
		→ 0.693147		→ 0.113706
2	0.693147		→ −0.193147	
		→ 0.5		
2	0.693147			

By theorem

$$P_4(x) = 0 + (x-1)1 + (x-1)^2(-0.306853) + (x-1)^2(x-2)(0.113706)$$

then Substuting $x = 1.5$ get

$$P_4(1.5) = 0.409074 \cong 0.405465$$

$$= \ln(1.5)$$

Other polynomail form and Conditions:

We have mentioned so for two polynomail forms the Lgrange form and Newton form.

There several other forms in Commone use Chiefly expansions in orthogonal polynomails.

Such expansions are the form

$$p = \sum_{i=1}^{n} a_i \, p_{i-1} \quad \cdots (1)$$

With (p_i) is a sequence polynomails satisfying three term recurrence $p_{-1}(x) = 0$, $p_0(x) = 1$

$$p_{i+1}(x) = A_i (x - B_i) p_i(x) - C_i p_{i-1}(x) \qquad i = 0, 1, 2, \cdots$$

for a Certion $A_0 \neq 0$, B_i, C_i

Remark: The Newton polynomail form fits this pattern (1) with $A_i = 1$, $B_i = x_{i+1}$, $C_i = 0$ for all i → Check

The Condition of the representation (1) in terms of the of the basis $(p_i)_{i=1}^{n} \in \mathbb{P}_n$ measures the possible relative change in p as a result of a unite change in the Cofficient vector $(a) = (a_i)$.

To be specifc, we measure the size of the polynomail p by the number

$$\|p\| = \max_{a \le x \le b} |p(x)|$$

Also, we measure the size of the vector $a = (a_j)$ by the number

$$\|a\| = \max_{1 \le i \le n} |a_i|$$

Then for all Coefficients vector a

$$m\|a\| \le \| \sum_i a_i P_{i-1} \| \le M \|a\|$$

With $$m = \min_a \frac{\| \sum_i a_i P_{i-1} \|}{\|a\|}$$

$$M = \max_a \frac{\| \sum_i a_i P_{i-1} \|}{\|a\|}$$

Def: The Condition number of the representation (1) is then defined as

$$Cond(P_i) = \frac{M}{m}$$

Remark: In Constructional the representation (1), we are bounded to make rounding error so that instead of the vector a, we actually produce the perturbed vector $a + \sigma a$

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Consequently we will not have the polynomial P, but the perturbed polynomial

$$P + \sigma P = \sum (a_i + \sigma a_i) P_{i-1}$$

Now by

$$m\|a\| \le \| \sum a_i P_{i-1} \| \le M \|a\|$$

$$\frac{m\|\sigma a\|}{M\|a\|} \le \frac{\|\sigma P\|}{\|P\|} \le \frac{M\|\sigma a\|}{m\|a\|}$$

Showing that a change in a of relative size $\frac{\|\sigma a\|}{\|a\|}$ may result in relative change $\frac{\|\sigma p\|}{\|p\|}$ in p as large as $\text{Cond}(p_i)$ times the relative chang in a and least $1/\text{Cond}(p_i)$ as large

Remark : In form (1) if $A_0 = 1$, $B_0 = 0$ $A_j = 2$

$B_j = 0$, $C_j = 1$

Then we have what is so called

Chebyshev Polynomail.

↳ H.W

الواجب

Chapter 2.

Limitations of polynomial approximation

recall that

$P_n g$ be polynomial such that $(P_n g)(\tau_i) = g(\tau_i)$

error

$$\|e_n\| = \|P_n g - g\| \leq \|(x-\tau_1)(x-\tau_2)\cdots(x-\tau_n)\| \, |[\tau_1, \tau_2, \ldots, \tau_n, x] g| \leq$$

$$\|(x-\tau_1)\cdots(x-\tau_n)\| \max_{a \leq x \leq b} \left| \frac{g^{(n)}(\xi)}{n!} \right|$$

Remark: This chapter focus:

I - Choice of interpolation point.

II - on the behaviour of g in $[a,b]$

Def: (Chebyshev points)

divided the semicircle into n equal arcs and then take their midpoints and project on $[a,b]$.

Remark: If the behaviour of the polynomial is bad then piecewise polynomial is a good approach.

I - Choice of $(\tau_i)_{i=1}^n$.

Runge example $g(x) = \frac{1}{1+25x^2}$, $[-1,1]$

Remark: Uniform spacing can have bad cancequances.

Proof See Runge example with uniform spacing and Chebyshev nodes.

II. Behaviour of $g(x)$

$$g(x) = \sqrt{|x|} \qquad x \in [-1, 1]$$

See page 22 كل دالة قابلة للاشتقاق تكون مستمرة
لكن ليس كل دالة مستمرة قابلة للاشتقاق مثل $|x|$ لذا أسامي.

Chapter 3 Piecewise Linear Approximation

Piecewise linear approximation may not the partical significance of cubic spline or even higher order, approximation.

But it shows most of the essential features of piecewise polynomial approximation in a simple and easily understandable setting.

Broken line interpolation:

We denote the piecewise linear or broken line interpolation to g at point $\tau_1, \tau_2, \dots, \tau_n$ with $a = \tau_1 < \tau_2 < \dots < \tau_n = b$ by $I_2 g$

the subscript 2 refers to the order of the polynomial pieces that make up the interpolant.

The interpolant is given by:

$$I_2 g(x) = g(\tau_i) + (x - \tau_i)[\tau_i, \tau_{i+1}]g$$
$$\text{on } \tau_i \le x \le \tau_{i+1}, \quad i = 1, 2, \dots, n-1$$

$$I_2 g(x) = g(\tau_1) + (x - \tau_1)[\tau_1, \tau_2]g$$
$$I_1 g(x) = g(\tau_1) + (x - \tau_1)[\tau_2, \tau_3]g$$

Since $g(x) = g(\tau_i) + (x - \tau_i)[\tau_i, \tau_{i+1}]g + (x - \tau_i)(x - \tau_{i-1})[\tau_i, \tau_{i+1}, x]g$

we have for $\tau_i \le x \le \tau_{i+1}$

$$g(x) - I_2 g(x) = (x - \tau_i)(x - \tau_{i+1})[\tau_i, \tau_{i+1}]g$$

Therefore with $\Delta\tau_i = \tau_{i+1} - \tau_i$ we get the error estimates:

$$|g(x) - I_2 g(x)| \le \left(\frac{\Delta\tau_i}{2}\right)^2 \max_{\tau_i \le \xi \le \tau_{i+1}} \left|\frac{\ddot{g}(\xi)}{2}\right|$$

is case g has two continuous derivatives, hence with

$$|\Gamma| = \max |\Delta\tau_i|$$

$$\|g - I_2 g\| \le \frac{1}{8} |\Gamma|^2 \|\ddot{g}\|$$

Remark: Now if $\tau \to 0$ we have error $\to 0$
ie. $I_2 g \to g$ and hence the broken line interpolation is nearly the best (optimal).

Broken line interpolation is nearly optimal

Let $\$_2$ denote the collection or linear space of all continuous broken lines on $[\tau_1, \dots, \tau_n]$ with broken $\tau_1, \dots, \tau_{n-1}$

Remark: The notation $\$_2$ reflects the fact that these are splines of order 2. (degree 1)

Lemma: If $f(x)$ is chosen to be the best linear approximation of g then $\|g - f\|_\infty \le \|g - I_2 g\|_\infty \le 2\|g - f\|_\infty$

proof First, we have

$$I_2 f = f \quad \dots (1)$$

Second

$$\|I_2 g\|_\infty = \|g\|_\infty \quad \dots (2)$$

$$\|I_2 g\|_\infty = \max |I_2 g(\tau_i)| = \max |g(\tau_i)| \le \|g\|_\infty$$

$I_2 g(\tau_i) = g(\tau_i)$

Now from (1) we have

$I_2 f = f \rightarrow I_2 f - f = 0$, therefore

$$\|g - I_2 g\|_\infty = \|g - I_2 g + I_2 f - f\|_\infty$$

$$= \|(g - f) - (I_2 g - I_2 f)\|_\infty$$

$$\leq \|g - f\|_\infty - \|I_2 g - I_2 f\|_\infty$$

$$\leq \|g - f\|_\infty + \|I_2 (g - f)\|_\infty$$

by eq. (2)

$$\leq \|g - f\|_\infty + \|g - f\|_\infty$$

$$\leq 2 \|g - f\|_\infty$$

It is clear that

$$\|g - f\|_\infty \leq \|g - I_2 g\|$$

Since f is best broken line interpolation

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$$\|I_2 f - I_2 g\| \leq \|I_2 (g - f)\|$$

Least square Approximation by broken lines :

We have discuss least-squares (or L_2) approximation by broken lines.

For this we need a convenient basis for $\$_2$

Let $\tau_0 = \tau_1$, $\tau_{n+1} = \tau_n$ and set

$$H_i(x) = \begin{cases} (x - \tau_{i-1}) / (\tau_i - \tau_{i-1}) & \tau_{i-1} \le x \le \tau_i \\ (\tau_{i+1} - x) / (\tau_{i+1} - \tau_i) & \tau_i \le x \le \tau_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

basis

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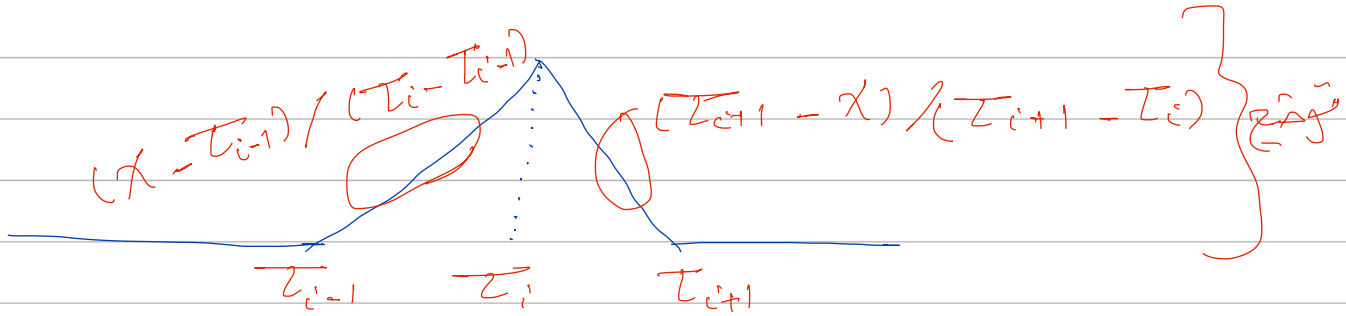

Thus functions have been called hat function

Clearly $H_i \in \$_2$ all i and is obvious that

$$H_i(\tau_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases} \quad \forall\, i, j \qquad (1)$$

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basis ← linear independent Span

Coefficient ≥ 0

Span ← linear Combination

$f \in \$_2 \rightarrow f = \sum c_i H_i$

This shows that $\sum_{i=1}^{n} g(\tau_i) H_i$ is an element of $\$_2$ that is a gree with g at $\tau_1, \tau_2, \ldots, \tau_n$

$= g(\tau_1) H_1(\tau_1) + g(\tau_2) H_2(\tau_1) + \cdots + g(\tau_n) H(\tau_1)$

While by $I_2 f$

Note: $I_2 g$ is the only element of $\$_2$ with that property.

This establishes the "Lagrange form"

$$I_2 g = \sum_{i=1}^{n} g(\tau_i) H_i(x) \quad \cdots (1)$$

for the broken line interpolation. It is also implies that $(H_i)_{i=1}^{n}$ is a basis for $\$_2$, so that every broken line on $[\tau_1, \ldots, \tau_n]$ with breaks $\tau_2, \ldots, \tau_{n-1}$ Can be written in exactly one Way as linear Combination of the H_i.

The Coordinates of a given $f \in \$_2$ with respact to the basis $H_i(x)$ Consist Simply of the values $(f(\tau_1), \ldots, f(\tau_n))$ at the breaks, that is

$$f = \sum_{i=1}^{n} f(\tau_i) H_i \quad \text{for all } f \in \$_2$$

Let $L_2 g$ be the least-squares approximation to g from $\$_2$ that is $L_2 g \in \$_2$ and

$$\int |g(x) - L_2 g(x)|^2 dx = \min_{f \in \$_2} \int |g(x) - f(x)|^2 dx$$

نبحث عن دالة f التي تجعل الفرق أقل ما يمكن

We determine $L_2 g$ with the aid of the normal equation that is we find a minimum of

$$\int \left| g(x) - \sum_{j=1}^{n} \alpha_j H_j(x) \right|^2 dx$$

By setting it's find partial derivitives with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$ to zero.

This gives the linear system $(M\alpha = b)$

$$\sum_{j=1}^{n} \left[\int H_i(x) H_j(x)\, dx \right] \alpha_j = \int H_j(x) g(x)\, dx \qquad i = 1, \ldots n \quad \ldots (3)$$

توضيح

$$\begin{cases} 2 \int \left[g(x) - \sum \alpha_j H_j(x)\, dx \right] \times H_i(x) = 0 \\ \text{متعامدة وخطي العمودي} \\ \int g(x) H_i(x) - \sum_{j=1}^{n} \int \alpha_j H_j H_i(x)\, dx = 0 \end{cases}$$

ننقل لهذا الطرف

for the Coffficient $(\alpha_i)_{i=1}^{n}$ of the broken line L_2-approximation to g.

$$L_2 g = \sum_{j=1}^{n} \alpha_j H_j$$

To find M

Since $\sum \alpha_j \left(\int_a^b H_i(x) H_j(x)\, dx \right) =$

$$\alpha_{i-1} \int_{\tau_{i-1}}^{\tau_i} H_i H_{i-1}\, dx + \alpha_i \int_{\tau_{i-1}}^{\tau_{i+1}} H_i H_i\, dx + \alpha_{i+1} \int_{\tau_i}^{\tau_{i+1}} H_i H_{i+1}\, dx$$

(1) (2) (3)

for (1) we have

$$\int_{\tau_{i-1}}^{\tau_i} H_i H_{i-1}\, dx = \int_{\tau_{i-1}}^{\tau_i} \frac{x - \tau_{i-1}}{x - \tau_{i-1}} \times \frac{\tau_i - x}{\tau_i - x}\, dx = \ldots = \frac{\tau_i - \tau_{i-1}}{6} = \frac{\Delta \tau_i}{6} \quad \ldots (4)$$

Similarly for (3)

$$\int_{\tau_i}^{\tau_{i+1}} H_i H_{i+1}\, dx = \cdots = \frac{\Delta\tau_i}{6} \quad \cdots (5)$$

H.W

While for (2) we have

$$\int_{\tau_{i-1}}^{\tau_{i+1}} H_i H_i\, dx = \int_{\tau_{i-1}}^{\tau_{i+1}} H_i^2\, dx = \int_{\tau_{i-1}}^{\tau_i} H_i^2\, dx + \int_{\tau_i}^{\tau_{i+1}} H_i^2\, dx = \cdots = \frac{\tau_{i+1} - \tau_{i-1}}{3} \quad \cdots (6)$$

i.e. M is tridiagonal

$$M = \left\{ \frac{\Delta\tau_i}{6}, \frac{\tau_{i+1} - \tau_{i-1}}{3}, \frac{\Delta\tau_i}{6} \right\}$$

$$\Delta\tau_0 = \Delta\tau_n = 0$$

Also, M is strictly diagonally domaint M is non-Singular

$M\alpha = b$ has a unige Solution.
We can take the least sequare approximation to $L_2 g$

$$L_2 g = \sum_{i=1}^{n} \alpha_i H_i(x)$$

Theorem: The L_2-approximation $L_2 g$ to $g \in C[a,b]$ that is to a continuous function g on $[a,b]$ by element of $\$_2$ satisfies $\|L_2 g\| \leq 3\|g\|$

Proof To bound $\|L_2 g\| = \max |(L_2 g)(x)|$

$= \max |\alpha_i H_i(\tau_i)| = \max |\alpha_i|$ in terms of $\|g\|$

Consider the equation

$$\frac{\Delta\tau_{i-1}}{6}\alpha_{i-1} + \frac{\tau_{i+1}-\tau_{i-1}}{3}\alpha_i + \frac{\Delta\tau_i}{6}\alpha_{i+1} = B_i = \int_a^b H_i(x) g(x)\,dx \quad \text{--- (1)}$$

multiply eq (1) by $\frac{6}{\tau_{i+1}-\tau_{i-1}}$ we get

$$\frac{\Delta\tau_{i-1}}{\tau_{i+1}-\tau_{i-1}}\alpha_{i-1} + 2\alpha_i + \frac{\Delta\tau_i}{\tau_{i+1}-\tau_{i-1}}\alpha_{i+1} = 3\hat{B}_i \quad \text{--- (2)}$$

where $\hat{B}_i = \int_a^b \hat{H}_i(x) g(x)\,dx$ and $\hat{H}_i(x) = \frac{2}{\tau_{i+1}-\tau_{i-1}} H_i(x)$

The system is strictly diagonally dominant uniformly in τ ($\tau_{i+1} - \tau_{i-1} = \Delta\tau_{i-1} + \Delta\tau_i$) then

$\therefore \max_i |\alpha_i| = \|\alpha\| = |\alpha_j|$ we get form eq (2)

$$|2\alpha_j| = \left|3\hat{B}_j - \frac{\alpha_{j-1}\Delta\tau_j + \alpha_{j+1}\Delta\tau_j}{\Delta\tau_{j-1} + \Delta\tau_j}\right| \leq 3|\hat{B}_i| + |\alpha_j|$$

$\therefore \|\alpha\| \leq 3|\hat{B}_j| \leq 3\|\hat{B}_j\|$

$\therefore B_i = \int_a^b \hat{H}_i\, g(x)\,dx$ and $\hat{H}_i(x) = \frac{2}{\tau_{i+1}-\tau_{i-1}} H_i(x)$

Postive for $\tau_{i-1} \leq x \leq \tau_{i+1}$ and zero otherwise and $\int_a^b \hat{H}_i(x)\,dx = 1$

$$\longrightarrow |\hat{B}_j| = \left|\int_a^b H_i(x) g(x)\,dx\right|$$

$$\leq \underbrace{\int_a^b H_i\,dx}_{\|g\|} \max\{|g(x)| : \tau_{i-1} \leq x \leq \tau_{i+1}\}$$

$$\longrightarrow \|L_2 g\| \leq 3\|g\|$$

Q/ prove that

$$\|g - I_2 g\| \leq \frac{1}{8} \left(\frac{b-a}{n-1}\right)^2 \|\acute{g}\| = O(n^{-2})$$

Proof

Chapter 4.

Piecewise Cubic Interpolation

Broken lines are neither very smooth nor very efficient approximation one has to go to piecewise polynomail approximation with higher order pieces.

The most popular choice continues to be a Cubic approximation function.

Given the data $g(\tau_1), \ldots, g(\tau_n)$ with $a = \tau_1 < \tau_2 < \ldots < \tau_n = b$, we interpolant f to g as follows:

on each interval $[\tau_i, \tau_{i+1}]$, we have f agree with Some polynomail of order 4,

(1) $f(x) = p_i(x)$, $\tau_{i-1} \le x \le \tau_{i+1}$ for some $p_i \in \mathbb{P}_{<4}$

$i = 1, 2, \ldots, n-1$

The ith polynomail piece p_i is mode to satisfy the Conditions

(2) $p(\tau_i) = g(\tau_i)$, $p(\tau_{i+1}) = g(\tau_{i+1})$

$p_i'(\tau_i) = s_i$, $p_{i+1}'(\tau_{i+1}) = s_{i+1}$ $\quad i = 1, 2, \ldots, n-1$

Here $s_1, s_2, \ldots s_n$ are free parametars.

The resulting piecewise cubic function.

f agree with g at $\tau_1, \ldots, \tau_n$ and is in $C^{(1)}[a,b]$ that is Continuous and has a Continuous first derivative on $[a,b]$.

In order to compute the coefficient of the ith polynomial piece p_i, we use it's Newton form.

$$p_i(x) = p_i(\tau_i) + (x-\tau_i)[\tau_i, \tau_i]p_i + (x-\tau_i)^1[\tau_i, \tau_i, \tau_{i+1}]p_i + (x-\tau_i)^2(x-\tau_i)[\tau_i, \tau_i, \tau_{i+1}, \tau_{i+1}]p_i$$

We determine it's coefficients form the divided difference table for p_i based on the data table in p. 40 (2)

τ $[\,]\rho$ $[\,,\,]\rho$ $[\,,\,,\,]\rho$ $[\,,\,,\,,\,]\rho$

τ_i $g(\tau_i)$

τ_i^- $g(\tau_i^-)$

s_i $\dfrac{[\tau_i, \tau_{i+1}]g - s_i}{\Delta\tau_i}$

$[\tau_i, \tau_{i+1}]g$ $\dfrac{s_{i+1} + s_i - 2[\tau_i, \tau_{i+1}]g}{(\Delta\tau_i)^2}$

τ_{i+1}^- $g(\tau_{i+1}^-)$ $\dfrac{s_{i+1} - [\tau_i, \tau_{i+1}]g}{\Delta\tau_i}$

s_{i+1}

τ_{i+1} $g(\tau_{i+1})$

This shows that in terms of the shifted powers

$$(x - \tau_i)^r$$

$$\rho_i(x) = c_{1,i} + c_{2,i}(x-\tau_i) + c_{3,i}(x-\tau_i)^2 + c_{4,i}(x-\tau_i)^3$$

With

$$\rho_i(\tau_i) = c_{1,i} = g(\tau_i)$$

$$\rho_i'(x) = c_{2,i} + 2c_{3,i}(x-\tau_i) + 3c_{4,i}(x-\tau_i)^2$$

$$\rho_i'(\tau_i) = c_{2,i} = s_i$$

$$c_{3,i} = \frac{\rho_i'(\tau_i)}{2}$$

$$= [\tau_i, \tau_i, \tau_{i+1}]\rho_i - \Delta\tau([\tau_i, \tau_i, \tau_{i+1}, \tau_{i+1}]\rho_i)$$

$$= [\tau_i, \tau_{i+1}]g - s_i / \Delta\tau_i - c_{4,i}\Delta\tau_i$$

$$C_{y,i} = \frac{\ddot{P}_i(\tau_j)}{6}$$

$$= S_i + S_{i+1} - 2[\tau_i \cdot \tau_{i+1}]_g / (\Delta\tau_i)^2$$

Different piecewise Cubic interpolation schemes differ in thechoice of the slopes $(S_i)_{i=1}^{n}$

We discuss the best known Choices:

1. piecewise Cubic Hermite interpolation.
2. piecewise Cubic Bessel interpolation.
3. Akima's interpolation.

Cubic Spline Interpolation

Def: Given a function $g(x)$ defined on $[a,b]$ and set of node points $a = x_1 < x_2 < \cdots < x_n = b$ a Cubic spline interpolation f of g is a function f that satisfying:

1- f Cubic polynomail denoted by S_j for g on each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, n-1$

2- $f(x_j) = g(x_j)$ $\quad j = 1, 2, \ldots, n-1$

3- $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ $\quad j = 1, 2, \ldots, n-2$

4- $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ $\quad j = 1, 2, \ldots, n-2$

5- $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$

6- Taking one set of the following boundary Conditions

a) $f''(x_1) = f''(x_n) = 0$ Natural Cubic spline

b) $f'(x_1) = g'(x_1)$
$f'(x_n) = g'(x_n)$ } Complete Cubic spline

First Approach

The Condition of the definition can be applied to Cubic polynomail

$$S_j(x) = a_j + b_j(x - \tau_j) + c_j(x - \tau_j)^2 + d_j(x - \tau_j)^3$$
$$j = 1, 2, \dots, n-1$$

It is clear form Condition (3) that

$$S_{j+1} = a_{j+1} + b_j(\tau_{j+1} - \tau_j) + c_j(\tau_{j+1} - \tau_j)^2 + d_j(\tau_{j+1} - \tau_j)^3$$

for simplicity, let $\Delta\tau_j = \tau_{j+1} - \tau_j$

$$a_{j+1} = a_j + b_j\Delta\tau_j + c_j(\Delta\tau_j)^2 + d_j(\Delta\tau_j)^3 \quad \dots (1)$$

$$j = 1, 2, \dots, n-1$$

Now using the Condition (4) and define

$b_n = f'(\tau_n)$ and hence

$$S_j'(x) = b_j + 2c_j(x - \tau_j) + 3d_j(x - \tau_j)^2$$

$$S_j'(x) = b_j \quad , j = 1, 2, \dots, n-1$$

Hence

$$b_{j+1} = b_j + 2c_j\Delta\tau_j + 3d_j(\Delta\tau_j)^2 \quad \dots (2)$$

Another relation between the coefficient of S_j can be obtained by defining

$$C_n = \frac{f''(\tau_n)}{2}$$ and apply condition (5) yields:

$$C_{j+1} = C_j + 3 d_j \Delta\tau_j \qquad j = 1, 2, \ldots n-1 \quad \rightsquigarrow (3)$$

Solving for d_j eq. (3) and substituting into eq. (1) and (2) have

$$d_j = \frac{C_{j+1} - C_j}{3 \Delta\tau_j} \quad \rightsquigarrow (4)$$

$$b_{j+1} = b_j + \Delta\tau_j (C_j + C_{j+1}) \quad \rightsquigarrow (5)$$

Solving eq. (1) for b_j we have

$$b_j = \frac{1}{\Delta\tau_j}(a_{j+1} - a_j) - \frac{\Delta\tau_j}{3}(2C_j + C_{j+1}) \rightsquigarrow (6)$$

Reducing the index by (1), we have:

$$\frac{1}{\Delta\tau_{j-1}}(a_j - a_{j-1}) - \frac{\Delta\tau_{j-1}}{3}(2C_{j-1} + C_j$$

Subsitituting the values into eq. (5) gives

$$\Delta\tau_{j-1}\, c_{j-1} + 2(\Delta\tau_{j-1} + \Delta\tau_j)c_j + \Delta\tau_j\, c_{j+1} = \frac{3}{\Delta\tau_j}(a_{j+1} - a_j) - \frac{3}{\Delta\tau_{j-1}}(a_j - a_{j-1}) \quad j = 1, \ldots, n-1 \quad (7)$$

This system involves the unknowns $\{c_j\}_{j=1}^{n-1}$ the using (6) and (4) to find $\{b_j\}_{j=1}^{n-1}$ and $\{d_j\}_{j=1}^{n-1}$ respectively.

"Second Approach"

In general التعريف

Def: A Spline function $S(x)$ of a degree k with n node $x_1 < x_2 < \dots < x_n$ has the properties:

i) $S(x)$ is given in the interval $[x_i, x_{i+1}]$ $i = 0, 1, \dots, n$ (where $x_0 = -\infty$, $x_n = \infty$) by a polynomail of degree at most k.

ii) $S(x)$ and all derivatives of orders $1, 2, \dots, k-1$ are Continuous on $(-\infty, \infty)$.

In the Case $k = 3$ the polynomails in each of the interval are (at most) Cubics and their first and Second derivatives are Continuous at the end points of the intervals.

Such a set of polynomail form a Cubic spline.

We now see how Such a set Can be Constructed Suppose that we are given a set of point $x_1 < x_2 < \dots < x_n$ not necessarily equally spaced, and a set of values $f(x_1), f(x_2), \dots, f(x_n)$ at these points.

Take a particular interval $[x_j, x_{i+1}]$ and fit a Cubic over the interval which satisfies the definition of the Cubic spline. Since the Cubic may differ form an interval to another.

Let the cubic be

$$F_i(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad x_i \le x \le x_{i+1} \quad \text{...(1)}$$

Equation (1) Contains 4 unknowns.

There are two obvious Conditions:

$$\left.\begin{aligned} F_i(x_i) &= f(x_i) \\ F_i(x_{i+1}) &= f(x_{i+1}) \end{aligned}\right\}$$ الشروط الأساسية

The remaining two Contitions are obtained by Choosing the Coefficients so that the first and second derivatives of $F(x_i)$ at x_0 are equal to the first and second derivatives of F_{i-1} at x_i. therefore

$$\left.\begin{aligned} F_i'(x_i) &= F_{i-1}'(x_i) \\ F_i''(x_i) &= F_{i-1}''(x_i) \end{aligned}\right\}$$ حتى أصبح الـ Smooth

Remark: The remain special problem at x_1 and x_n but we deal with these later.

نتعامل بتحسين وبلا x_n ، x_1 في تعدد نوع الـ Cubic

The Conditions are now Sufficient to determine the (n-1) cubics which Collectively Constitute the Cubic Spline $S(x)$

عدد القطع (n-1)

لكل نزة قطعة

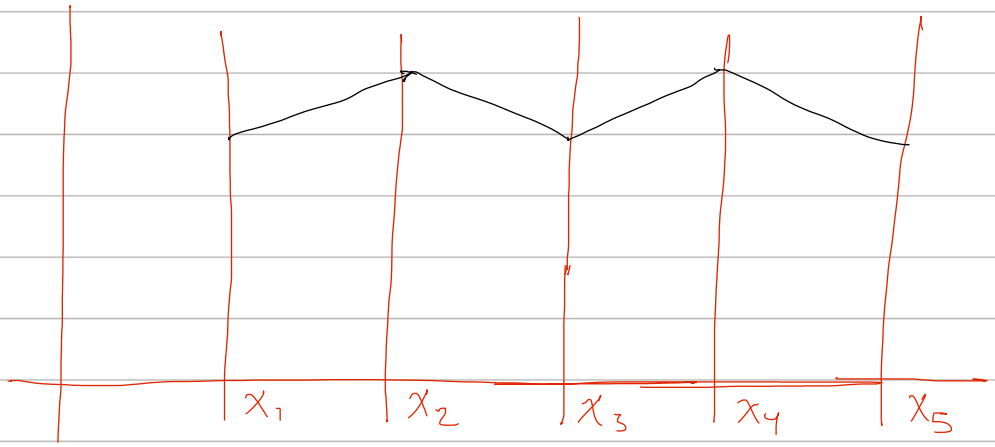
$$S(x) = F_i(x) \qquad x_i \le x \le x_{i+1}$$

عبارة عن مجموع القطع

Q/ How can solve thus equation?

* The simplest method this to not that "S(x) is linear in x and is also continuous over the whole interval $[x_1, x_n]$ it is not however differentiable. (متصلة ، linear مستقيم)

The graph of "S(x) is therefore of the type:



"S(x) can therefore be represented in $[x_i, x_{i+1}]$ by a linear function which is easily seen to be

$$"S(x) = "S(x_i) + \frac{x - x_i}{x_{i+1} - x_i} ("S(x_{i+1}) - "S(x_i)) \quad \sim (2$$

معادلة مستقيم تمر بنقطة

If we rewrite equation (2) in the form

$$"S(x) = \frac{x - x_i}{x_{i+1} - x_i} "S(x_{i+1}) - \frac{x - x_i}{x_{i+1} - x_i} "S(x_i)$$ and integrate

twise then, putting $\Delta x_i = x_{i+1} - x_i$ the result can be written

تكاملها مرتين واكتبها بالصيغة التالية
لـ

$$S(x) = \frac{(x-x_i)^3}{6\Delta x_i} S''(x_{i+1}) - \frac{(x-x_{i+1})^3}{6\Delta x_i} S''(x_i) + a(x-x_i) - b(x-x_{i+1}) \quad \text{---(3)}$$

ثابت التكامل

Remark: Any expression of the form $Ax+B$ may be written as $a(x-x_i) - b(x-x_{i+1})$ for suitable choice of a, b provided $x_i \neq x_{i+1}$

اي ثابت تكامل اكبر او اكثر يمكن كتابته بصيغة $a(x-x_i) - b(x-x_{i+1})$ على شرط

We now impose the conditions that $S(x_i) = f(x_i)$ and $S(x_{i+1}) = f(x_{i+1})$ so that form equation (3)

اذا وضعنا في (3) اول x يساوي و الثاني يساوي

$$f(x_i) = \frac{-(x_i - x_{i+1})}{6\Delta x_i} S''(x_i) + b(x_i - x_{i+1})$$

or

$$b = \frac{-f(x_i)}{\Delta x_i} - \frac{\Delta x_i}{6} S''(x_i) \quad \text{---(4)}$$

and

$$f(x_{i+1}) = \frac{(\Delta x_i)^3}{6\Delta x_i} S''(x_{i+1}) + a\,\Delta x_i$$

or

$$a = \frac{f(x_{i+1})}{\Delta x_i} - \frac{\Delta x_i}{6} S''(x_{i+1}) \quad \text{---(5)}$$

Subsituting eq. (4) and (5) in (3) gives after slight re-arrangement of terms: تبسيطات متنوعة

$$S(x) = \frac{S''(x)}{6}\left\{\frac{(x_{i+1}-x)^3}{\Delta x_i} - \Delta x_i (x_{i+1}-x)\right\} + \frac{S''(x_{i+1})}{6}\left\{\frac{(x-x_i)^3}{\Delta x_i} - \Delta x_i (x-x_i)\right\} +$$

$$f(x_i)\left[\frac{x_{i+1}-x}{\Delta x_i}\right] + f(x_{i+1})\left[\frac{x-x_i}{\Delta x_i}\right] \quad \text{---(6)}$$

صلح

the expression being valid for the interval $x_i \leq x \leq x_{i+1}$.

Differentiating equation (6) and putting $x = x_i$ we have:

$$S'(x) = \frac{S''(x_i)}{6}\left\{-\frac{3(x_{i+1}-x_i)^2}{\Delta x_i} + \Delta x_i\right\} - \frac{S''(x_{i+1})\Delta x_i}{6} - \frac{f(x_i)}{\Delta x_i} + \frac{f(x_{i+1})}{\Delta x_i} \quad \text{---(7)}$$

If we now replace i by $i-1$ in (6), differentiate and again put $x = x_i$, we obtain: نغير i بـ i-1

$$S'(x) = \frac{S''(x_i)}{6}\left\{-\frac{3(x_i - x_{i-1})}{\Delta x_{i-1}} - \Delta x_{i-1}\right\} + S''(x_{i-1})\Delta x_{i-1} - \frac{f(x_{i-1})}{\Delta x_{i-1}} + \frac{f(x_i)}{\Delta x_{i-1}} \quad \text{---(8)}$$

The Continuity $S'(x)$ at x_i now requires that the expressions on the right of (7) and (8) are equal and this leads to equation:

$$S''(x_{i-1})\Delta x_{i-1} + S''(x_i)(2(x_{i+1}-x_i)) + S''(x_{i+1})\Delta x_i = 6\left\{\frac{f(x_{i+1})-f(x_i)}{\Delta x_i} - \frac{f(x_i)-f(x_{i-1})}{\Delta x_{i-1}}\right\} \quad \text{---(9)}$$

Coefficient ... Loi

The set of $(n-1)$ equations equation (9) Contain $(n+1)$ unknown $S''(x_j)$ $j = 0, 1, \ldots, n$ and order to obtain a unique Solution, we must impose Conditions on $S''(x_0)$ and this usually done by taking the Spline in the intervals $[-\infty, x_0]$ and $[x_n, \infty]$ to be a straight line so that $S''(x_0) = S''(x_n) = 0$.

The special $S(x)$ So-determined is Called natural Cubic Spline.

Given these extra two Conditions the equations (9) are now Sufficient to determine "S(x_0) and so S(x).

Note: Node $x_0, \dots, x_n$
unknown $S(x_0), \dots, S(x_n)$
عدد الكل $n+1$

Gusse طريقة

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1^{(k)} = \frac{1}{a_{11}} \left[b_1 - a_{12}x_2^{(k-1)} - \dots - a_{1n}x_n^{(k-1)} \right]$$

$$x_2^{(k)} = \frac{1}{a_{22}} \left[b_2 - a_{21}x_1^{(k-1)} - \dots - a_{2n}x_n^{(k-1)} \right]$$

$$\vdots$$

$$x_n^{(k)} = \frac{1}{a_{nn}} \left[b_n - a_{n1}x_1^{(k-1)} - \dots - a_{n,n-1}x_{n-1}^{(k-1)} \right]$$

$$x^0 = (x_1^0 \quad x_2^0 \quad \dots \quad x_n^0)$$
$$= (0 \quad 0 \quad \dots \quad 0)$$

$$\| x^{(k)} - x^{(k-1)} \| \leq \epsilon$$

The system of linear equations is of tri-diagonal form and such system can be solved either by direct methods such as Guss-elemination or if n is large by indirect method like Guss-SRidel

Ex: Fit a natural Cubic spline to data below and use it to estimate $f(55)$

x	$f(x)$
25	5
36	6
49	7
64	8
81	9

$S''(25) = S''(81) = 0$

$S''(36), S''(49), S''(64)$ → unknown

Sol We use eq (9) to form a set linear equations for $S''(36), S''(49), S''(64)$ and we take $S''(25) = S''(81) = 0$

$i = 1$

$$0.11 + 2(24)S''(36) + 13S''(49) = 6\left\{\frac{1}{13} - \frac{1}{11}\right\}$$

$i = 2$

$$13.S''(36) + 2(28)S''(49) + 15S''(64) = 6\left\{\frac{1}{15} - \frac{1}{13}\right\}$$

$i = 3$

$$15S''(49) + 2(32)S''(64) + 17.0 = 6\left\{\frac{1}{17} - \frac{1}{15}\right\}$$

Rearranging these equation in a form suitable for Guss-Sedial iteration they become

$$''S(36) = \frac{1}{48}\left\{0.083916 + 13\left[''S(49)\right]\right\}$$

$$''S(49) = \frac{1}{56}\left\{0.061538 + 13\,''S(36) + 15\,''S(64)\right\}$$

$$''S(64) = \frac{1}{64}\left\{0.047059 + 15\,''S(49)\right\}$$

Starting at $''S(36) = ''S(49) = ''S(64) = 0$ and iterating produces the values

$$''S(36) = -0.001594$$

$$''S(49) = -0.000568$$

$$''S(64) = -0.000602$$

The point at which we wish to estimate $x = 55$ lines in the interval $[49, 64]$ so we must use the cubic appropriate to that interval.

i.e. we use eq (6) with $x_{i+1} = 64$, $x_i = 49$ $x = 55$ and we obtain:

$$S(55) = \frac{''S(49)}{6}\left\{\frac{9^3}{15} - 15\times 9\right\} + \frac{''S(64)}{6}\left\{\frac{6^3}{15} - 15\times 6\right\} + 7\left[\frac{9}{15}\right] + 8\left[\frac{6}{15}\right]$$

$$= 0.008179 + 0.007585 + 7.4 = 7.415764 \text{ So}$$

$$f(55) \simeq 7.415764 \quad , \sqrt{55} = 7.416198$$

Ch. 5

Best Approximation properties of Complete Cubic Spline Interpolation and Its Error

(*) we shall that in this Chapter that $f = I_4 g$ to g minimizes $\int_a^b [\tilde{f}(x)]^2 dx$.

Lemma :- If g is twice differentiable, then the second derivative of the interpolation error $e = g - I_4 g$ is orthogonal to $\$_2$ that is $\int_a^b \tilde{e}(x)\, \phi(x)\, dx = 0 \quad \forall \phi \in \$_2$

Proof H.W

Theorem:- (Pythagoras) For every twice continuously differentiable function g on $[a,b]$

$$\int_a^b [g''(x)]^2 dx = \int_a^b [(I_4 g)''(x)]^2 dx + \int_a^b [(g - I_4 g)''(x)]^2 dx$$

Proof

$$g - I_4 g = e$$

$$g'' = (I_4 g + e)''$$

$$[g'']^2 = [(I_4 g)'' + e'']^2$$

$$\int_a^b [g'']^2 dx = \int_a^b [(I_4 g)''(x) + e''(x)]^2 dx$$

$$= \int_a^b [(I_4 g)]^2 dx + 2\underbrace{\int_a^b (I_4 g)''(x)\, e''\, dx}_{=0} + \int_a^b [e''(x)]^2 dx$$

But $\int_a^b (I_4 g)''(x)\, e''(x) = 0$ by lemma

Since $(I_4 g)'' \in \$_2 = 0$ ∎

Corollary: Among all functions f with two continuous derivatives that agree with the function g at the nodes $\tau_0, \tau_1, \dots, \tau_{n+1}$, the function $I_4 g$ uniquely minimizes $\int_a^b [f''(t)]^2 dt$

Proof

$$\int_a^b [f''(t)]^2 dt = \int_a^b [(I_4 g)''(t)]^2 dt +$$

$$\int_a^b [f''(t) - (I_4 g)'']^2 dt$$

$$\geq \int_a^b [(I_4 g)'']^2 dt$$

Corollary :- The second derivative of the Complete Cubic spline interpolant to the function g is the least squares approximation from the space Φ_2 of broken lines to the second derivative of g that is

$$(I_4 g)'' = L_2(g'')$$

Proof :- $I_4 f = f$ and so $(I_4 f)'' = f'' = S$

$$I_4 f - f = 0$$

Now $\int_a^b [g'' - S(x)]^2 dx =$

$$\int_a^b [(I_4 g)'' - S(x)]^2 + \int_a^b [g - I_4 g)'']^2 dx$$

$$\geq \int_a^b [g''(x) + (I_4 g)''(x)]^2 dx$$

with equality if and only if

$(I_4 g)'' = S$ Hence $L_2 g'' = I_4 g''$

$\blacksquare$

(2) Another Approach

Recall that

$$\int_a^b e'' H_i(x)\,dx = 0$$

$$\int_a^b [\ddot{g} - (I_4 g)'']^2 H_v(x)\,dx = 0$$

$$\int_a^b [\ddot{g} - (\sum_{j=1}^{n} \alpha_j H_j(x)] H_v(x)\,dx$$

$$\sum_{j=1}^{n} \int_a^b \alpha_j (H_i \cdot H_j)\,dx = \int_a^b \ddot{g}\, H_i(x)\,dx \quad \rightsquigarrow (1)$$

Since we have proved previously that

$$\sum_{j=1}^{n} \alpha_j \int_a^b (H_i \cdot H_j)\,dx = \int_a^b f H_i(x)\,dx \quad \rightsquigarrow (2)$$

Which has the least squares approximation

$$L_2 f = \sum_{j=1}^{n} \alpha_j H_j$$

from (2) and (1), we have $L_2 \ddot{g} = \sum \alpha_j H_j$ — (3)

Since we have

$$(I_4 g)'' = \sum \alpha_j H_j(x) \quad \rightsquigarrow (4)$$

Therefore from eq. (3) & (4) we have

$$(I_4 g)'' = L_2 \ddot{g} \quad \text{XX}$$

Theorem : $\|I_4 g''\| \leq 3\|\ddot{g}\|$

↳ H.W

Lemma : $\|\ddot{e}\| \leq \frac{1}{2}|\tau|^2 \|\ddot{g}''\|$

↳ H.W

Corollary : For a twice Continuously differentiable function g,

$$\|g - I_4 g\| \leq \frac{1}{2}|\tau|^2 \operatorname{dist}(\ddot{g}, \$_2)$$

↳ H.W

