# Al-Nahrain University

## Collage of Science

Department of Mathematics and Computer Applications


# Foundation of Mathematics/Year One


**Dr. Aamena Rasim**




***<u>BOOKS:</u>***


1. ***Foundation of Mathematics, Part 1, by Hadi Jaber, Read Shaker Naoum and Nader Gorge Mansor.*** (Arabic)
2. ***Fundamental Concepts of Modern Mathematics, by Max D. Larsen.***
3. ***Schaum's Outline of Set Theory and Related Topics (Schaum's Outline Series)***.

# Chapter 1

# Set Theory

## 1.1 Sets

A **set** is unordered a collection of abstract objects, which are referred to as the **elements** or **members** of the set. We write $a \in A$ to denote that $a$ is an element of the set $A$ (and $b \notin A$ to denote that $b$ is not a member of the set $A$). Braces (curly brackets) $\{\cdots\}$ are used to denote a set.

**Example 1.1.1.**

- $S = \{0, 1, 2, 3, 5, 6, 8\}$
- $A = \{7, p, -3, Baghdad\}$
- $B = \{0, 1, \{3, 4\}\}$

Here, for example, $0 \in S$, $p \in A$ but $2 \notin B$.

Note that the elements of a set do not have to be "the same sort of thing", and can even themselves be sets (e.g. one of the members of $B$ is the set $\{3, 4\}$).

**The Empty Set**

The **empty set** (or occasionally **null set**), denote by $\emptyset$, has no element.

Note that $\{\emptyset\}$ is not the empty set; rather, it is the set containing the single element $\emptyset$.

**Example 1.1.2.**

- If $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$, then the set containing those elements which are in both $A$ and $B$ is $\emptyset$.
- The set containing positive integers which are less than 1 is $\emptyset$.
- The set of months of the year beginning with the letter "W" is $\emptyset$.

**Infinite set (in brief)**

So far we have seen examples of **finite sets**, so called since they contain finitely many elements. On the first problems sheet you will see examples of **infinite sets** (which contain *infinitely* many elements), such as

- $\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$ is the set of **natural numbers**;
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots\}$ is the set of **integers**;
- $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$ consists of all fractions, and is the set of **rational numbers** (or **rationals**);
- $\mathbb{R}$, the set of **real numbers**, consists of everything on the ”number line” (i.e. the rational plus the *irrational*).

**Set terminology and notation**

Let $X$ and $Y$ be sets. Then we say that

- $X$ is a **subset** of $Y$, denoted $X \subseteq Y$, if every element of $X$ is also an element of $Y$.
- $X$ and $Y$ are **equal**, denoted $X = Y$, if they contain exactly the same elements.
- $X$ is a **proper subset** of $Y$, denoted $X \subset Y$, if $X \subseteq Y$ and $x \neq Y$.

**Example 1.1.3.**

Suppose that $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 4\}$ and $D = \{2, 1, 3\}$. Then

$B \subset A$, $C \subseteq A$, $C \subset D$, $B \subseteq D$, $D \subset B$, $B = D$, etc.

**Note** that one way of establishing that $X = Y$ is to show that *both* $X \subseteq Y$ and $Y \subseteq X$.

**Set Operations**

Let $A, B$ be sets. Then

- The **union** $A \cup B$ of $A$ and $B$ is the set of all elements which are elements of $A$ **or** elements of $B$ *(or BOTH)*. i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The **intersection** $A \cap B$ of $A$ and $B$ is the set of all elements which are elements of $A$ **and** elements of $B$. i.e. $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- The **difference** $A \setminus B$ is the set of all elements of $A$ which are **not** elements of $B$. i.e. $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.
- If the intersection of two sets $A$, $B$ is the empty set (i.e. $A \cap B = \emptyset$) then we say that $A$ and $B$ are **disjoint**.

**Example 1.1.4.** Let $A = \{1, 2\}$ and $B = \{2, 4, 5, 6\}$. Then $A \cup B = \{1, 2, 4, 5, 6\}$, $A \cap B = \{2\}$, $A \backslash B = \{1\}$, $B \backslash A = \{4, 5, 6\}$.

**Remark 1.1.5.** Let $A$ and $B$ any two sets. It is clear that

**(i.)** $A \subseteq A \cup B$ and $A \subseteq B \cup A$.

**(ii.)** $A \cap B \subseteq A$ and $B \cap A \subseteq A$.

## 1.2 Algebra of Sets

**Theorem 1.2.1.** *Let $A$, $B$ and $C$ be any three sets with $U$ is the universal set. Then*

***(1)*** $A \cap U = A$, $A \cup \emptyset = A$ *(**Identity**)*.

***(2)*** $A \cup U = U$, $A \cap \emptyset = \emptyset$ *(**Domination**)*.

***(3)*** $A \cup A = A$, $A \cap A = A$ *(**Idempotent Laws**)*.

***(4)*** $(A^c)^c = A$, $A \cup A^c = U$, $A \cap A^c = \emptyset$, $U^c = \emptyset$, $\emptyset^c = U$ *(**Complement Laws** )*.

***(5)*** $A \cup B = B \cup A$, $A \cap B = B \cap A$ *(**Commutative Laws**)*.

***(6)*** $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ *(**Associative Laws**)*.

***(7)*** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *(**Distributive Laws**)*.

***(8)*** $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ *(**De Morgan's Laws**)*.

***(9)*** $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$ *(**Absorption Laws**)*.

*Proof.*

**(4)** We want to prove that $(A^c)^c = A$, we have to show

- $(A^c)^c \subseteq A$ and
- $A \subseteq (A^c)^c$

Let $x \in (A^c)^c$.

From the definition of the complement (page 3) we get: $x \in U$ and $x \notin A^c$. Thus $x \in U$ and $x \in A$, this implies that $x \in A$.

Hence $(A^c)^c \subseteq A$ - - - - - - - - - - - - (*)

Let $x \in A$. Therefore $x \in U$ and $x \notin A^c$. From the definition of the complement (page 3) we get $x \in (A^c)^c$.

Hence $A \subseteq (A^c)^c$. - - - - - - - - - - - - - - (**)

Therefore from (*) and (**) we get $(A^c)^c = A$.

**(5)** We want to prove that $A \cup B = B \cup A$, we have to show

- $A \cup B \subseteq B \cup A$ and
- $B \cup A \subseteq A \cup B$

Let $x \in A \cup B$.

From the definition of the union (page 2) we get: $x \in A$ or $x \in B$. That is $x \in B$ or $x \in A$ this implies that $x \in B \cup A$.

Hence $A \cup B \subseteq B \cup A$ - - - - - - - - - - - - - (*)

Let $x \in B \cup A$. From the definition of the union (page 2) we get: $x \in B$ or $x \in A$. That is $x \in A$ or $x \in B$, this implies that $x \in A \cup B$.

Hence $B \cup A \subseteq A \cup B$ - - - - - - - - - - - - - - (**)

Therefore from (*) and (**) we get $A \cup B = B \cup A$.

**(7)** $A \cup (B \cap C) = \{x : x \in A \text{ or } x \in (B \cap C)\}$

$$= \{x : x \in A \text{ or } x \in B \text{ and } x \in C\}$$

(note that if $x \in A$ then $x \in (A \cup B)$ and $x \in (A \cup C)$, i.e. $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$. Also if $x \in (B \cap C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$, i.e. $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$). Thus

$$A \cup (B \cap C) = \{x : x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C\},$$

hence

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

**(8)** $(A \cup B)^c = \{x : x \in U \text{ and } x \notin (A \cup B)\}$

$$\begin{aligned}
&= \{x : x \notin A \text{ and } x \notin B\} \\
&= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\} \\
&= \{x : x \in A^c \text{ and } x \in B^c\} \\
&= A^c \cap B^c.
\end{aligned}$$

**(9)** $A \cap (A \cup B) = \{x : x \in A \text{ and } x \in (A \cup B)\}$

$$\begin{aligned}
&= \{x : x \in A \text{ and } x \in A \text{ or } x \in B\} \\
&= A.
\end{aligned}$$

□

**Example 1.2.2.** Let $A$, $B$ and $C$ be any sets. Prove that

**(1)** $A \backslash A = \emptyset$.

**(2)** $(A \backslash B) \cap B = \emptyset$.

**(3)** $(B \backslash A) = B \cap A^c$.

**(4)** $(A \backslash C) \cap (C \backslash B) = \emptyset$.

**(5)** $(A \cup (B \cap C))^c = (C^c \cup B^c) \cap A^c$.

**(6)** If $A \subset B$ then $A \cap B = A$.

*Proof.*

**(1)** Let $x \in A \backslash A$. Then $x \in A$ and $x \notin A$ which is impossible. Hence $A \backslash A$ must be empty.

**(2)**
$$\begin{aligned}(A\backslash B) \cap B &= \{x : x \in B \text{ and } x \in A\backslash B\} \\ &= \{x : x \in B \text{ and } x \in A \text{ and } x \notin B\} \\ &= \emptyset.\end{aligned}$$

**(3)**
$$\begin{aligned}(B\backslash A) &= \{x : x \in B \text{ and } x \notin A\} \\ &= \{x : x \in B \text{ and } x \in A^c\} \\ &= B \cap A^c.\end{aligned}$$

**(4)**
$$\begin{aligned}(A\backslash C) \cap (C\backslash B) &= (A \cap C^c) \cap (C \cap B^c) && \text{(By using 3)} \\ &= (C \cap C^c) \cap (A \cap B^c) && \text{(By using the associative and the commutative laws)} \\ &= \emptyset \cap (A \cap B^c) \\ &= \emptyset.\end{aligned}$$

**(5)**
$$\begin{aligned}(A \cup (B \cap C))^c &= A^c \cap (B \cap C)^c && \text{(By using the second De Morgan law)} \\ &= (B^c \cup C^c) \cap A^c && \text{(By using the first De Morgan law)} \\ &= (C^c \cup B^c) \cap A^c && \text{(By using the commutative law).}\end{aligned}$$

**(6)**

□

## 1.3 Cartesian Product of Sets

**Definition 1.3.1.** Let $A, B$ and $C$ be any sets. The **product set** of $A$ and $B$, written $A \times B$, consists of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$, i.e.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

This definition can be extended to more than two sets, e.g.

$$A \times B \times C = \{(a, b, c) : a \in A \text{ and } b \in B \text{ and } c \in C\}.$$

The product of sets with themselves have a special notation:
$A \times A = A^2, A \times A \times A = A^3$, etc.

**Example 1.3.2.** Let $A = \{0, 1\}$, $B = \{a, b\}$. Find $A \times B$ and $A^2$

**(1)** $A \times B = \{(0, a), (0, b), (1, a), (1, b)\}$.

**(2)** $A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

**Example 1.3.3.** Let $S = \{1, 2, 3\}$, $T = \{1, 5\}$. Find $S \times T$, $T \times S$, $S^2$, $T^2$, $(S \times T) \times S$ and $S \times (T \times S)$.

**(1)** $S \times T = \{(1, 1), (1, 5), (2, 1), (2, 5), (3, 1), (3, 5)\}$.

**(2)** $T \times S = \{(1, 1), (1, 2), (1, 3), (5, 1), (5, 2), (5, 3)\}$.

**(3)** $S^2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

**(4)** $T^2 = \{(1, 1), (1, 5), (5, 1), (5, 5)\}$.

**(5)** $(S \times T) \times S = \{(1, 1), (1, 5), (2, 1), (2, 5), (3, 1), (3, 5)\} \times S$.

$= \{((1, 1), 1), ((1, 1), 2), ((1, 1), 3), ((1, 5), 1), ((1, 5), 2), ((1, 5), 3), ((2, 1), 1), ((2, 1), 2), ((2, 1), 3),$
$((2, 5), 1), ((2, 5), 2), ((2, 5), 3), ((3, 1), 1), ((3, 1), 2), ((3, 1), 3), ((3, 5), 1), ((3, 5), 2), ((3, 5), 3)\}$

**(6)** $S \times (T \times S) = S \times \{(1, 1), (1, 2), (1, 3), (5, 1), (5, 2), (5, 3)\}$.

$$= \{(1, (1, 1)), \cdots, (2, (1, 1)), \cdots, (3, (1, 1)) \cdots, (3, (5, 3))\}.$$

**Remark 1.3.4.**

- From the previous example we noted that $S \times T \neq T \times S$ in general.
- From the previous example we noted that $(S \times T) \times S \neq S \times (T \times S)$ in general.
- For each real numbers $a, b, c$ and $d$, if $(a, b) = (c, d)$ then $a = c$ and $b = d$.

**Lemma 1.3.5.** *Let $A, B$ and $C$ be any sets. Then*

***(1)*** $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

***(2)*** $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

*Proof.*

**(1)** $A \times (B \cup C) = \{(x, y) : x \in A \text{ and } y \in (B \cup C)\}$

$= \{(x, y) : x \in A \text{ and } (y \in B \text{ or } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}$

$= (A \times B) \cup (A \times C)$.

**(2)** $A \times (B \cap C) = \{(x, y) : x \in A \text{ and } y \in (B \cap C)\}$

$=$

$=$

$=$ □

**Example 1.3.6.** *Let* $A = \{a, b\}$, $B = \{2, 3\}$ *and* $C = \{3, 4\}$. *Find* $A \times (B \cap C)$ *and* $(A \times B) \cap (A \times C)$.

## 1.4 Power Set

**Definition 1.4.1.** Let $A$ be any set. The **power set** of A, denoted by $\mathcal{P}(A)$ or $2^A$, is the class of all subsets of $A$. In particular, if $A = \{a, b, c\}$, then

$$\mathcal{P}(A) = \{A, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

In general, if $A$ is finite, say $A$ has $n$ elements, then $\mathcal{P}(A)$ will have $2^n$ elements.

**Example 1.4.2.** Let $A = \{0, 9\}$ and $B = \{a, b, 1, 2\}$. Find $\mathcal{P}(A)$ and $\mathcal{P}(B)$.

**Definition 1.4.3.** Let $A$ be a non empty set. Suppose that $\Psi = \{A_1, A_2, \cdots, A_n\}$ is a set contains subsets of $A$, that is $A_1 \subset A$, $A_2 \subset A, \cdots, A_n \subset A$. Then $\Psi$ is called a **partition** if $A$ if:

**(i.)** For each $a \in A$, $a$ belongs to some member of $\Psi$.

**(ii.)** The members of $\Psi$ are disjoint, that is $A_i \cap A_j = \emptyset$ for $i \neq j$.

**Example 1.4.4.** Let $A = \{1, 2, 3, 4\}$ and $\Psi = \{\{1\}, \{4\}, \{2, 3\}\}$. Does $\Psi$ is a partition for $A$? Why?

$\Psi$ forms a partition for $A$ because

* Each element of $A$ belongs to some member of $\Psi$.

* The members of $\Psi$ are disjoint, that is $\{1\} \cap \{4\} = \emptyset$, $\{1\} \cap \{2, 3\} = \emptyset$ and $\{4\} \cap \{2, 3\} = \emptyset$.

We end this chapter by the following definitions.

**Definition 1.4.5** (Open and Closed intervals)**.**

Let $a, b$ are real numbers. The sets of the form

$$(a, b) := \{x : x \in \mathbb{R} \text{ and } a < x < b\},$$

$$[a, b] := \{x : x \in \mathbb{R} \text{ and } a \leq x <\leq b\},$$

$$[a, b) := \{x : x \in \mathbb{R} \text{ and } a \leq x < b\},$$

$$(a, b] := \{x : x \in \mathbb{R} \text{ and } a < x \leq b\},$$

are called **intervals**. The interval $(a, b)$ is called **open**, the interval $[a, b]$ is called **closed** and the intervals $[a, b), (a, b]$ are called **half-open** of **half-closed**.

**Definition 1.4.6** (Open and Closed intervals)**.**

Let $a \in \mathbb{R}$. We define the sets

$$[a, +\infty) := \{x : x \in \mathbb{R} \text{ and } x \geq a\},$$

$$(a, +\infty) := \{x : x \in \mathbb{R} \text{ and } x > a\},$$

$$(-\infty, a] := \{x : x \in \mathbb{R} \text{ and } x \leq a\},$$

$$(-\infty, a) := \{x : x \in \mathbb{R} \text{ and } x < a\}.$$

The sets $[a, +\infty), (a, +\infty), (-\infty, a]$ and $(-\infty, a)$ are called **unbounded intervals** or **rays** in $\mathbb{R}$. In the same manner, we may also use the notation

$$(-\infty, +\infty) := \mathbb{R}$$

which allows us to think of $\mathbb{R}$ as an unbounded interval "from both sides ".

**Remark 1.4.7.** The **SYMBOLS** "$-\infty$" and "$+\infty$" are called **minus infinity** and **plus infinity** respectively. Be careful in that they **ARE NOT REAL NUMBERS**, **but just a convenient notation** which should be understood as that "there is no end ". Sometimes the " $+$ " might be omitted in the front of "$+\infty$", and we may write merely "$\infty$" instead.

Chapter 2

# Mathematical Logic

## 2.1 Statements

**Definition 2.1.1.** A statements or proposition is a declarative statement (i.e. a statement that declares something is a fact), that is either true, or false, but not both. For example

• Baghdad is the capital of Iraq.

• $2 + 3 = 6$.

<u>**Notations**</u>

• Questions and instructions are **NOT** statements, since we can not assign them a "truth value"

• We use lower case letters (usually $p, q, r, s$ etc.) to represent statements.

• The **truth value** of a statement denoted by either $T$ for true or $F$ for false.

### 2.1.1 Combining

We use three main operations to combine statements into **compound statements**:

• negation denoted by "$\sim$".

• conjunction denoted by "$\wedge$".

• disjunction denoted by "$\vee$".

<u>**NEGATION**</u>

The **negation** of a statement $p$ is denoted by $\sim p$, and can be read as "not $p$" or "it is not the case the $p$". The statement $\sim p$ is false when $p$ is true, and true when $p$ is false. For example, if

• $p$: Today is Friday

• $q$: There are at least ten people in the room.

• $r$: For all $x \in \mathbb{Z}, x > 0$.

Then

• $\sim p$: Today is not Friday

• $\sim q$: There are fewer than ten people in the room.

• $\sim r$: There exists $x \in \mathbb{Z}, x \leq 0$.

## CONJUNCTION

Let $p$ and $q$ be statements. Then the **conjunction** of $p$ and $q$ is denoted by $p \wedge q$, and can be read as "$p$ and $q$". $p \wedge q$ is the statement which is true when both $p$ and $q$ are true, and false otherwise. For example, if

• $p$: Today is Friday.

• $q$: There are at least ten people in the room.

• $r$: $x > 3$.

• $s$: $x = 5$.

Then

• $p \wedge q$: Today is Friday and there are at least ten people in the room.

• $\sim p \wedge q$: Today is not Friday and there are at least ten people in the room.

• $\sim r \wedge \sim s$: $x \leq 3$ and $x \neq 5$

**Remark 2.1.2** (Use of brackets)**.** To be clear about the meaning of compound statements we will usually use brackets, **except** in the case of negation: the negation symbol only applies to the symbol (or combination of symbols) immediately to its right. So, $\sim p \wedge q$ means $(\sim p) \wedge q$, not $\sim (p \wedge q)$. For example, if:

• $r$: $x > 3$.

• $s$: $x = 5$.

Then

• $\sim r \wedge s$: $x \leq 3$ and $x = 5$.

• $\sim (r \wedge s)$: not the case that ($x > 3$ and $x = 5$).

## DISJUNCTION

Let $p$ and $q$ be statements. Then the **disjunction** of $p$ and $q$ is denoted by $p \vee q$, and can be read as "$p$ or $q$". $p \vee q$ is the statement which is true when either $p$ or $q$ or both are true, and false otherwise.

For example, if

- $r$: $x = 2 + y$.
- $s$: $y = -1$.

Then

- $r \vee s$: either $x = 2 + y$ or $y = -1$ or both.
- $s \vee \sim r$: either $y = -1$ or $x \neq 2 + y$ or both.
- $r \vee \sim s$: either $x = 2 + y$ or $y \neq -1$ or both.

**TRUTH TABLES**

It is often useful to represent statements in a **truth table**. **T** is used to indicate that a statement is true and **F** to indicate that it is false. We create a column for each statement (e.g. $p$ and $q$), and new columns for each compound statement of interest. For example:

| $p$ | $q$ | $\sim p$ | $p \wedge q$ | $p \vee q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | F |

**Example 2.1.3.** Complete the truth table below.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \wedge q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $p \vee \sim q$ |
|---|---|---|---|---|---|---|---|
| T | | | | | | | |
| T | | | | | | | |
| F | | | | | | | |
| F | | | | | | | |

## 2.2 The conditional statements

Let $p$ and $q$ be statements. The **conditional statement** $p \to q$ is the statement "if $p$, then $q$". Here $p$ is the **antecedent** and $q$ is the **consequent**. Note that we can alternatively "pronounce" $p \to q$ as

- $p$ implies $q$
- $p$ is a sufficient condition for $q$
- $q$ is a necessary condition for $p$
- $p$ only if $q$

Note that $p \to q$ is **false** when $p$ is true and $q$ is false, and **true** otherwise.

**Example 2.2.1.** Let

- $p$: it is raining • $q$: 90% of students attend lecturers

then

- $p \to q$: if it is raining then 90% of students attend lectures.
- $q \to p$: if 90% of students attend lectures, then it is raining.

**Note that $p \to q$ is not the same as $q \to p$.**

**Example 2.2.2.** *Let*

- $p$: $\sqrt{x} > 1$ • $q$: $x = 4$.

then

- $q \to p$: if $x = 4$ then $\sqrt{x} > 1$.
- $p \to q$: if $\sqrt{x} > 1$ then $x = 4$.

**The biconditional statements**

Let $p$ and $q$ be statements. The **biconditional statement** $p \longleftrightarrow q$ is the statement "$p$ if and only if $q$". $p \longleftrightarrow q$ is true when $p$ and $q$ have the same **truth** value, and false otherwise.

**Example 2.2.3.** • $p$: it is raining • $q$: 90% of students attend lecturers

then

- $p \longleftrightarrow q$: it is raining if and only if 90% of students attend lectures.
- $q \longleftrightarrow p$: 90% of students attend lectures if and only if it is raining.

**Example 2.2.4.** • $p$: $n$ is odd number • $q$: $n^2$ is odd number.

then

- $q \longleftrightarrow p$: $n$ is odd if and only if $n^2$ is odd number.
- $p \longleftrightarrow q$: if $n^2$ is odd number if and only if $n$ is odd number.

- $p$: $n$ is even number • $q$: $n^2$ is odd number.

then

- $q \longleftrightarrow p$: $n$ is even if and only if $n^2$ is odd number.
- $p \longleftrightarrow q$: $n^2$ is odd number if and only if $n$ is even number.

**Truth tables for conditional statements**

| $p$ | $q$ | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ | $\sim p$ | $\sim q$ | $\sim p \to \sim q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | T |
| T | F | F | T | F | F | T | F |
| F | T | T | F | F | T | F | T |
| F | F | T | T | T | T | T | T |

**Example 2.2.5** (Translating sentences)**.** Translate each of the following statements:

**1.** You can access the website only if you pay the fee.

The condition statement $p$ only if $q$.

p: "You can access the website"

q: "You pay the fee"

then the statement translates as $p \to q$.

**2.** You can access the website from university only if you are studying mathematics or you are not first year student.

p: "You can access the website from university ".

q: "You are studying mathematics ".

r: "You are a first year student ".

The require statement is $p \to (q \vee \sim r)$.

**Theorem 2.2.6** (Algebra of statements)**.** *Let p, q and r be the three statements. Then:*

*(1)* $p \wedge q = q \wedge p$ *and* $p \vee q = q \vee p$. ***(Commutative Laws)***.

*(2)* $p \wedge p = p$ *and* $p \vee p = p$ ***(Idempotent Laws)***.

*(3)* $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ *and* $(p \vee q) \vee r = p \vee (q \vee r)$ ***(Associative Laws)***.

*(4)* $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ *and* $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ ***(Distributive Laws)***.

*(5)* $\sim (p \wedge q) = (\sim p) \vee (\sim q)$ *and* $\sim (p \vee q) = (\sim p) \wedge (\sim q)$ ***(De Morgan's Laws)***.

*(6)* ***The Identity Laws***

$p \wedge F = F$, $p \wedge T = p$, $p \vee T = T$ *and* $p \vee F = p$.

*(7)* ***The Complementarity***

$p \wedge (\sim p) = F$, $p \vee (\sim p) = T$ *and* $\sim (\sim p) = p$.

**Definition 2.2.7** (Tautology)**.**

A compound statement that is always true, no matter what truth values the proposition is made up of, is called a **tautology**. For example, the statement $p \vee \sim p$ is a tautology, as shown in the following truth table:

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

**Example 2.2.8.** Show that which of the following statements is a tautology:

1. $[p \wedge (p \rightarrow q)] \rightarrow q$.
2. $\sim (p \wedge q) \longleftrightarrow (\sim p \vee \sim q)$.
3. $(p \rightarrow q) \longleftrightarrow (p \wedge \sim q)$.
4. $(p \rightarrow q) \rightarrow (q \rightarrow p)$.

<u>Solution 1:</u>

| $p$ | $q$ | $p \rightarrow q$ | $p \wedge (p \rightarrow q)$ | $p \wedge (p \rightarrow q) \rightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | **T** |
| T | F | F | F | **T** |
| F | T | T | F | **T** |
| F | F | T | F | **T** |

Since the last column (column 5) is all true then the statement $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

<u>Solution 2:</u>

| $p$ | $q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p$ | $\sim q$ | $(\sim p \vee \sim q)$ | $\sim (p \wedge q) \longleftrightarrow (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | **T** |
| T | F | F | T | F | T | T | **T** |
| F | T | F | T | T | F | T | **T** |
| F | F | F | T | T | T | T | **T** |

Since the last column (column 8) is all true then the statement is a tautology.

<u>Solution 3:</u>

Solution 4:

**Theorem 2.2.9.** *Let $p$, $q$ and $r$ be statements. Then $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.*

*Proof.* Homework

$\square$

**Definition 2.2.10** (Contradiction)**.**
A compound statement that always false, no matter what truth values the statement is made up of,is called a **contradiction**. For example, the statement $p \wedge \sim p$ is a contradiction, as shown in the following truth table:

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

**Remark 2.2.11.** The compound statement $Q$ is a contradiction if an only is $\sim Q$ is a tautology. For example, we know that $p \vee \sim p$ is a tautology thus $\sim (p \vee \sim p)$ is a contradiction, as shown in the following truth table:

| $p$ | $\sim p$ | $p \vee \sim p$ | $\sim (p \vee \sim p)$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

**Example 2.2.12.** Show that the following compound statement, $((p \to q) \wedge p) \wedge \sim q$, is a contradiction.

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge p$ | $\sim q$ | $((p \to q) \wedge p) \wedge \sim q$ |
|---|---|---|---|---|---|
| T | T | T | T | F | **F** |
| T | F | F | F | T | **F** |
| F | T | T | F | F | **F** |
| F | F | T | F | T | **F** |

Since the last column (column 6) is all false then the statement is a contradiction.

**Definition 2.2.13** (Logical Equivalence)**.**
Let $P$ and $Q$ be compound statements. Then $P$ is said to be **equivalence** $Q$ ($P \equiv Q$) if and only if the *truth* table for $P$ is the same as the *truth* table for $Q$ **or** if $P \longleftrightarrow Q$ is a tautology.

**Example 2.2.14.** Let P : $p \to q$ and Q : $\sim p \vee q$. Show that $P \equiv Q$.

| $p$ | $q$ | $p \to q$ | $\sim p$ | $\sim p \vee q$ |
|---|---|---|---|---|
| T | T | **T** | F | **T** |
| T | F | **F** | F | **F** |
| F | T | **T** | T | **T** |
| F | F | **T** | T | **T** |

Since column 3 and column 5 are the same then $P$ and $Q$ are equivalence.

**Remark 2.2.15.**

1. For any statement $p$ we have:
   - $p \equiv p$.
   - $p \vee p \equiv p$.
   - $p \wedge p \equiv p$.
2. $(p \longleftrightarrow q) \equiv [(p \longrightarrow q) \wedge (q \longrightarrow p)]$.

*Proof.* Homework

□

**Example 2.2.16.** Are the following compound statements are logically equivalent?

$P : p \vee (q \wedge r)$ and $Q : (p \vee q) \wedge (p \vee r)$.

Solution

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | | | | | |
| T | T | F | | | | | |
| T | F | T | | | | | |
| T | F | F | | | | | |
| F | T | T | | | | | |
| F | T | F | | | | | |
| F | F | T | | | | | |
| F | F | F | | | | | |

**Definition 2.2.17** (Converse and Inverse)**.**

The **converse** of the statement $p \longrightarrow q$ is $p \longrightarrow p$.

The **inverse** of the statement $p \longrightarrow q$ is $\sim p \longrightarrow \sim p$.

Note that Neither of these is equivalent to the original statement $p \longrightarrow q$. Check with truth table.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $q \longrightarrow p$ | $\sim p \longrightarrow \sim q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | T | T |
| F | T | T | F | T | F | F |
| F | F | T | T | T | T | T |

However the statements $q \longrightarrow p$ and $\sim p \longrightarrow \sim q$ are equivalent, which brings us to the next definition.

**Definition 2.2.18** (Contrapositive)**.**

The contrapositive of the statement $p \longrightarrow q$ is $\sim q \longrightarrow \sim p$. The original statement $p \longrightarrow q$ and the contrapositive $\sim q \longrightarrow \sim p$ are logically equivalent:

Note that the contrapositive of the statement $q \longrightarrow p$ is $\sim p \longrightarrow \sim q$, so that the converse and the inverse are logically equivalent to one another.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \rightarrow q$ | $\sim q \longrightarrow \sim p$ |
|---|---|---|---|---|---|
| T | T | F | F | **T** | **T** |
| T | F | F | T | **F** | **F** |
| F | T | T | F | **T** | **T** |
| F | F | T | T | **T** | **T** |

**Example 2.2.19.**

1. Equilateral triangle is isosceles triangle.
2. non isosceles triangle is not equilateral triangle.

The first compound statement is $p \longrightarrow q$, so the second compound statement is $\sim q \longrightarrow \sim p$.

## 2.3 Quantifiers

Two fundamental kinds of **quantification** in mathematics logic are **universal quantification** and **existential quantification**. The traditional symbol for the **universal quantifier** "all" is "$\forall$", a rotated letter "$A$", and for the **existential quantifier** "exists" is "$\exists$", a rotated letter "$E$".

**Definition 2.3.1** (The Universal Quantifier).
The expression $\forall x\ P(x)$, denotes the **universal quantification** of the statement formula $P(x)$, translate into English language, the expression is understood as "for all $x$, $P(x)$ holds", "for each $x$, $P(x)$ holds" or "for every $x$, $P(x)$ holds ". The symbol "$\forall$" is called the **universal quantification** and $\forall x$ means all the object $x$ in the **universal**. If this is followed by $P(x)$, then the meaning is that $P(x)$ is true for every object $x$ in the **universal**. For example:

$P : \forall\ n \in \mathbb{N}, n > -2.$

$P$ is a universal quantifier and it is true.

$Q : \forall\ x \in \mathbb{R}, x > 1.$

$Q$ is a universal quantifier and it is false.

**Definition 2.3.2** (The Existential Quantifier).
The expression $\exists\ x\ P(x)$, denotes the **existential quantification** of the statement formula $P(x)$, translated into English language, the expression is understood as "There exists an $x$ such that $P(x)$" or "There is at least one $x$, such that $P(x)$". The symbol "$\exists$" is called the **existential quantifier** and $\exists\ x$ means at least one object $x$ in the **universal**. If this is followed by $P(x)$, then the meaning is that $P(x)$ is true for at least one object $x$ of the **universal**. For example:

$P : \exists\ n \in \mathbb{N}, 3n + 1 > 2.$

$P$ is an existential quantifier and it is true, because $n = 1$ satisfies $P$.

$Q : \exists\ x \in \mathbb{R}, x^2 + 1 = 0.$

$Q$ is an existential quantifier and it is false, because there is no real number that satisfies $Q$.

**Remark 2.3.3.**

There may be in one statement one or more universal quantifiers or one or more existential quantifiers. For example:

$P : \forall\ x \in A, \forall\ y \in \mathbb{N}, \forall\ z \in B,\ p(x, y, z).$

$Q : \forall\ x \in A, \exists\ y \in \mathbb{N},\ p(x, y).$

**Example 2.3.4.** Let $A = \{-1, 0, 1\}$. Then

$P :\ \forall\ x \in A,\ \exists\ y \in A,\ x + y = 0$, is true statement.

$Q :\ \exists\ x \in A,\ \forall\ y \in A,\ x + y = 0$, is false statement.

From the previous example we note that

$$\exists\ y\ \forall x,\quad p(x, y) \neq \forall\ x\ \exists\ y, p(x, y).$$

**<u>Negation of Statement has Quantifier</u>**

Suppose that we have the following statement:

"Every student in this class has average score eighty ".

The negation of this statement if:

"Not true that every student has average score eighty ".

This means that there exists at least one student in this class his average score not eighty.

If $M$ is the set of all students in the class and $p(x)$ is the statement "has average score eighty" then we can translate statements as follows:

$Q :\ \forall\ x \in M,\ p(x)$ and

$\sim (Q) :\sim (\ \forall\ x \in M,\ p(x)) \equiv \exists\ x \in M,\ \sim p(x).$

Therefore and in general if $p(x)$ is a statement depends on $x$ and defined on a set $A$, then

**Theorem 2.3.5.**

***1.*** $\sim (\ \forall\ x \in A,\ p(x)) \equiv \exists\ x \in A,\ \sim p(x).$

***2.*** $\sim (\ \exists\ x \in A,\ p(x)) \equiv \forall\ x \in A,\ \sim p(x).$

*Proof.*

**1.** We will prove that $\sim (\ \forall\ x\ ,\ p(x))$ and $\exists\ x\ ,\ \sim p(x)$ are logically equivalent by showing that they are both true and they are both false.

Suppose that $\sim (\ \forall\ x\ ,\ p(x))$ is true, then $\forall\ x\ ,\ p(x)$ is false. This means that there exists $b \in U$, (where $U$ is the universal set), such that $p(b)$ is false. Hence $\sim p(b)$ is true, this implies the $\exists\ x,\ \sim p(x)$ is true.

Now suppose that $\sim (\ \forall\ x,\ p(x))$ is false, then $\forall\ x,\ p(x)$ is true. This means that for all $b \in U$, (where $U$ is the universal set), such that $p(b)$ is true. Hence $\sim p(b)$ is false, this implies the $\exists\ x,\ \sim p(x)$ is false.

Therefore

$$\sim (\ \forall\ x\ ,\ p(x)) \equiv \exists\ x\ ,\ \sim p(x).$$

**2.** By using (1) we get:

$$\begin{aligned}\sim (\ \forall\ x\ ,\ \sim p(x)) &\equiv \exists\ x\ ,\ \sim\sim p(x)\\ &\equiv \exists\ x\ ,\ p(x).\end{aligned}$$

Thus

$$\forall\ x\ ,\ \sim p(x) \equiv \sim (\ \exists\ x\ ,\ p(x)).$$

□

**Remark 2.3.6.** ***Note that to negate a quantified expression do the following:***

- ***Change the quantifier***
- ***Negate the predicate expression that follows the quantifier.***

***In general***

$$\boxed{\begin{array}{l}\sim (p \vee q) \equiv \sim p \wedge \sim q\\ \sim (p \wedge q) \equiv \sim p \vee \sim q\\ \sim (p \to q) \equiv p \wedge \sim q\end{array}}$$

**Example 2.3.7.** Find

(1) $\sim (\exists\ x \in \mathbb{R},\ \forall\ y \in \mathbb{R},\ x + y = y)$.

(2) $\sim (\forall\ x,\ \forall\ y\ \ \exists\ z,\ x + y + z = 18)$.

(3) $\exists\ x,\ [p(x) \to q(x)]$.

Solution(1):

$$\begin{aligned}\sim (\exists\ x \in \mathbb{R},\ \forall\ y \in \mathbb{R},\ x + y = y) &\equiv \forall\ x \in \mathbb{R}, \sim (\ \forall\ y \in \mathbb{R},\ x + y = y)\\ &\equiv \forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R},\ \sim (x + y = y)\\ &\equiv \forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R},\ x + y \neq y.\end{aligned}$$

Solution(2):

$$\begin{aligned}\sim (\forall\ x,\ \forall\ y\ \ \exists\ z,\ x + y + z = 18) &\equiv \exists\ x,\ \sim (\forall\ y\ \ \exists\ z,\ x + y + z = 18)\\ &\equiv \exists\ x,\ \exists\ y\ \ \sim (\ \exists\ z,\ x + y + z = 18)\\ &\equiv \exists\ x,\ \exists\ y\ \ \forall\ z,\ \sim (x + y + z = 18)\\ &\equiv \exists\ x,\ \exists\ y\ \ \forall\ z,\ x + y + z \neq 18.\end{aligned}$$

Solution(3):

**Example 2.3.8.** Negate the following statements:

(1) For all $n \in \mathbb{N}, 2n + 3 > 7$.

(2) There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ $xy \leq 2$.

Solution(1):

The statement in symbols is $\forall\ n \in \mathbb{N},\ 2n + 3 > 7$. Thus

$\sim (\forall\ n \in \mathbb{N},\ 2n + 3 > 7) \equiv \exists\ n \in \mathbb{N},\ 2n + 3 \leq 7.$

Solution(2):

The statement in symbols is $\exists\ y \in \mathbb{R}, \forall x \in \mathbb{R}\ xy \leq 2$. Thus

$\sim (\exists\ y \in \mathbb{R}, \forall x \in \mathbb{R},\ xy \leq 2) \equiv\ \forall\ y \in \mathbb{R}, \exists x \in \mathbb{R},\ xy > 2.$

## 2.4 Mathematical Proof

**Definition 2.4.1** (Logical Reasoning)**.**

Let $p_1, p_2, \cdots p_n$ be statements. Suppose that $p$ is a new statement can be inferred from $p_1, p_2, \cdots p_n$. The compound statement

$$p \text{ can be inferred from } p_1, p_2, \cdots p_n$$

is called ***argument***, $p_1, p_2, \cdots p_n$ called ***premisses*** and $p$ is called ***conclusion***. We will use the symbol

$$p_1, p_2, \cdots, p_n \;\vdash\quad p$$

An argument is said to be valid if the conclusion must be true whenever the premises are all true. An argument is **invalid** if it is not valid. It is possible for all the premises to be true and the conclusion to be false.

**Example 2.4.2.**

**Remark 2.4.3.** The argument

$$p_1, p_2, \cdots p_n \;\vdash\quad p$$

is true if and only if

$$p_1 \wedge p_2, \wedge \cdots \wedge p_n \;\rightarrow\; p$$

is a tautology

**Definition 2.4.4** (Mathematical Proof)**.** Let $p_1, p_2, \cdots p_n$ be statements. Suppose that $p$ is a new statement can be inferred from $p_1, p_2, \cdots p_n$. If the argument

$$p_1, p_2, \cdots p_n \;\vdash\quad p$$

is true then it called a **Mathematical proof**.

**Proof the conditional statements** $(p \rightarrow q)$

There are two procedures to prove statements $(p \rightarrow q)$.

**(1)** One of the most important ideas to understand is the method of conditional proof. The basic idea is to assume that $p$ is true and deduce that $q$ must be true. We suppose that $p$ is true, then using this with some known theorems and axioms to get $q$. In particular when we get $q$ in this way we prove that $p \to q$ is true. In this way we prove that $q$ is true when $p$ is true. Thus if $p_1, p_2, \cdots, p_n$ are known theorems and axioms, then to prove $p \to q$ we prove that

$$p, p_1, p_2, \cdots, p_n \vdash - \quad q$$

is a true argument.

**Example 2.4.5.** *Prove that*

*$a$ is even number $\longrightarrow a^2$ is even number.*

*Proof.* Suppose that $a$ is even number.

Then there exists $k \in \mathbb{N}$ such that $a = 2k$.

Thus $a^2 = 4k^2$.

That is $a^2 = 2(2k^2)$.

Since $2k^2 \in \mathbb{N}$, then $a^2$ is even number. $\square$

**(2)** Using the contrapositive. We can prove $p \to q$ by proving $\sim q \to \sim p$ because $p \to q \equiv \sim q \to \sim p$.

**Example 2.4.6.** Prove that $a^2$ is even number $\longrightarrow a$ is even number.

*Proof.* Note that

$p : a^2$ is even, $q : a^2$ is even. Thus to prove $p \to q$ using the contrapositive we will prove that

$p \to q \equiv \sim q \to \sim p$ where

$\sim q : a$ is even and $\sim p : a^2$ is even.

Suppose that $\sim q$ is true, that is $a$ is odd number.

Thus that exists $k \in \mathbb{N}$ such that $a = 2k + 1$

Hence, $a^2 = (2k+1)^2$.

This means that $a^2 = 4k^2 + 4k + 1$.

That is $a^2 = 2(2k^2 + 2k) + 1$

Since $2k^2 + 2k \in \mathbb{N}$, then $a^2$ is even number. $\square$

Note that, we proved that ($a$ is odd $\longrightarrow a^2$ is odd) that is we proved $\sim q \to \sim p$ is true, thus $p \to q$ is true which means that $a^2$ even $\longrightarrow a$ even.

**Proof statements of type ($p \leftrightarrow q$)**

There are three ways to prove $p \leftrightarrow q$ :

**(1)** We know that $p \longleftrightarrow q \equiv p \to q \wedge q \to p$. Thus to prove $p \longleftrightarrow q$, we first prove that $p \to q$ then we prove $q \to p$.

**(2)** We first prove that $p \to q$ and then prove $\sim p \to \sim q$ instead of $q \to p$.

**(3)** Prove $p \leftrightarrow q$ using equivalence statements as follows:

$p \leftrightarrow p_1$

$p_1 \leftrightarrow p_2$

$p_2 \leftrightarrow p_3$

$\vdots$

$p_n \leftrightarrow q$

**Example 2.4.7.** *Prove that $a$ is odd $\leftrightarrow a^2$ is odd.*

*Proof.* $a$ is odd $\leftrightarrow a = 2k+1$ for some $k \in \mathbb{N}$

$a = 2k+1 \leftrightarrow a^2 = 4k^2 + 4k + 1$

$a^2 = 2(2k^2 + 2k) + 1 \leftrightarrow a^2$ is odd number. □

**Prove Statements have Quantifier**

To prove statement $(\forall x, p(x))$, we suppose that $x$ is an element in the universal set then we prove $p(x)$ is true. This proves that $(\forall x, p(x))$ is true.

To prove statement $(\exists x, p(x))$, we suppose that there exists $x$ in $U$ (the universal set) which make $p(x)$ is true.

**Example 2.4.8.** Prove that for every real positive number $x$, $x + \frac{4}{x} \geq 4$

*Proof.* Let $x \in \mathbb{R}$ and $x > 0$. Then $(x-2)^2 \geq 0$ since the square of a real number is never negative. Expanding gives $x^2 - 4x + 4 \geq 0$. By assumption $x > 0$, so dividing by $x$ preserves the inequality and gives $x - 4 + \frac{4}{x} \geq 0$. Finally, adding 4 to both sides gives $x + \frac{4}{x} \geq 4$. □

**Example 2.4.9.** Prove that there exist integers $m$ and $n$ such that $2m + 3n = 12$.

*Proof.* Set $m = 3$ and $n = 2$. Then $2m + 3n = 2(3) + 3(2) = 6 + 6 = 12$. □

**Prove Statements ($p \vee r \to q$)**

To prove statements of this type we need to prove $p \to q$ and $r \to q$, this means that $q$ can be achieved from $r$ or $p$.

**Example 2.4.10.** *Prove that if $a = 0$ or $b = 0$ then $ab = 0$.*

*Proof.* We want to prove $a = 0 \vee b = 0 \to ab = 0$.

Suppose that $a = 0$, then $ab = (0)b = 0$,

thus $a = 0 \to ab = 0$.

Suppose that $b = 0$, then $ab = a(0) = 0$,

hence $b = 0 \to ab = 0$.

Therefore $a = 0 \vee b = 0 \to ab = 0$. □

**Example 2.4.11.** If $a$ is an even integer and $b$ is an odd integer, then $a + b$ is an odd integer.

*Proof.* Let $a$ be an even number and $b$ be an odd number. Then there exists $n, m \in \mathbb{N}$ such that

$$a = 2n \quad \text{and} \quad b = 2m + 1.$$

Thus $a + b = 2n + 2m + 1 = 2(n + m) + 1$. Hence $a + b$ is odd number. □

**Example 2.4.12.** For any two sets $A$ and $B$, prove that $A \cap B \subseteq A \cup B$.

*Proof.*

□

## Proof by Contradiction

We know that the contradiction is always false statement. For example $p \wedge \sim p$ is always false. To prove a statement $p$ by contradiction, we suppose that $\sim p$, then we try to find $r \wedge \sim r$ where $r$ is a compound statement contains $p$ or known theorems or axioms. That is $[\sim p \wedge (r \wedge \sim r)] \to p$.

Using the contradiction we can also prove statements : $p \to q$, $\forall x, p(x)$ and $\exists x, p(x)$. For example, to prove $p \to q$ using contradiction we:

(1) Suppose that $\sim (p \to q)$ is true. Since $\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$, thus our assumption is $p \wedge \sim q$ is true. This implies that $p$ is true and $\sim q$ is true.

(2) Try to get a contradiction to prove that $\sim (p \to q)$ is false, hence $(p \to q)$ is true.

**Example 2.4.13.** Prove that $x \neq 0 \to \frac{1}{x} \neq 0$.

*Proof.* $p : x \neq 0$ and $q : \frac{1}{x} \neq 0$

To prove $p \to q$, suppose that $\sim (p \to q)$ is true.

Since $\sim (p \to q) \equiv p \wedge \sim q$, then $p \wedge \sim q$ is true.

That is $x \neq 0 \wedge \frac{1}{x} = 0$ is true.

But $x\frac{1}{x} = 1$,

and since $\frac{1}{x} = 0$ then $x\frac{1}{x} = x(0) = 0$.

Thus $1 = 0$ which is a contradiction.

Hence $\sim (p \to q)$ is false and thus $p \to q$ is true. □

**Example 2.4.14.** Let $A$ be any set. Prove that $\emptyset \subseteq A$.

*Proof.* We want to prove that $(\forall x)(x \in \emptyset \to x \in A)$. We will use the contrapositive to prove $x \in \emptyset \to x \in A$. That is we will prove that

$$(\forall x)(x \notin A \to x \notin \emptyset).$$

Let $x$ be arbitrary (any element in $U$). Suppose that $x \notin A$, then it is clear that $x \notin \emptyset$ since $\emptyset$ is empty. Thus $x \notin A \to x \notin \emptyset$ is true, hence $x \in \emptyset \to x \in A$ is true. Therefore $\emptyset \subseteq A$. □

**<u>Solution Set</u>**

**Definition 2.4.15.** Let $p(x)$ be a statement in $x$ defined on a set $A$. Suppose that $a \in A$, if $p(a)$ is true then we say that $a$ is a solution for $p(x)$. The set of all $a$ that make $p(x)$ true called the **solution set** of $p(x)$. That is

$$S = \{a \in A : p(a) \text{is true}\}$$

**Example 2.4.16.** Let $A = \{0, 1, 2, 3\}$ and $2 - x \geq 1$ be a statement in $x$ defined on $A$. Then

$$S = \{0, 1\}.$$

**Example 2.4.17.** Find the solution set for all the following statements:

(1) $x - 2 < 5$, defined on $\{0, 1, 2, 3\}$.

(2) $|x| + 1 < 3$ defined on $\{0, 1, 2, 3, 5\}$.

(3) $(x - 1)(x + 2) = 0$ defined on $\{5, 6, 8\}$.

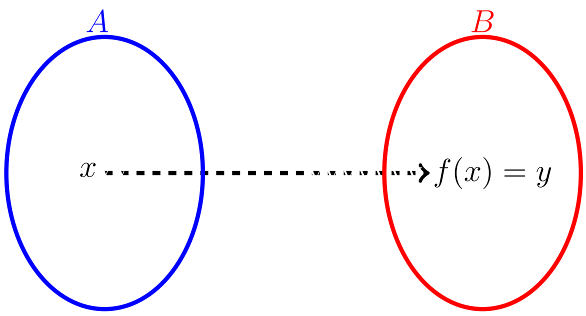
(4) $x^2 + 1 = 0$ defined on $\mathbb{R}$ (the set of real numbers).

# Chapter 3

# Mapping

## 3.1 Concepts and Definition

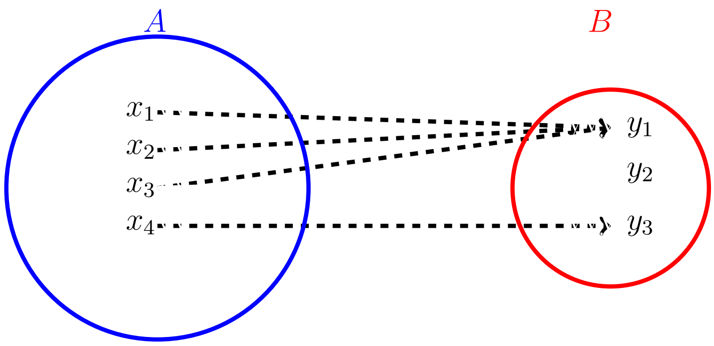
**Definition 3.1.1.** Let $A$ and $B$ be non-empty sets. A **function** (mapping) from $A$ to $B$ is a relation (formula, rule or correspondence) that assigns exactly one element of $B$ to each element of $A$.



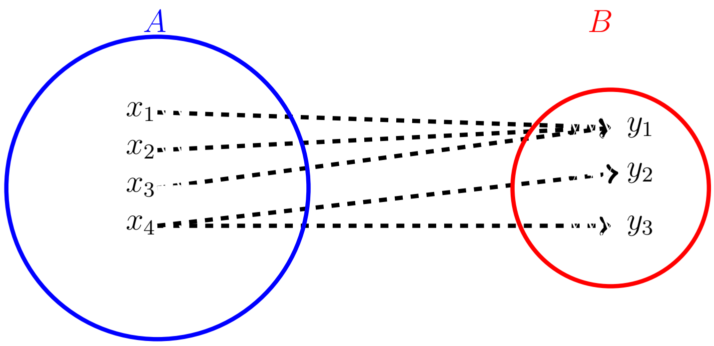

- Functions are often denoted by letters, such as $f$, $g$, $\cdots$, and so on.
- $A$ is the **domain** of $f$ and $B$ is the **codomain** of $F$.
- We write $f : A \to B$, and for $x \in A$, the element that assigns to $x$ is denoted by $f(x)$ it is an element in $B$, that is $f(x) = y$, $b \in B$. $y$ is the **image** of $x$ and $x$ is the **pre-image** of $y$.
- The subset of $B$ containing all images under $f$ is called the **range** of $f$.

**Example 3.1.2.**

1. The assignment $f : \{a, b, c\} \to \{0, 1\}$ given by $f(a) = 0$, $f(b) = 1$ and $f(c) = 1$ is a function.
2. The assignment $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(x) = 2x$ is a function.
3. Are these functions:




Each element of $A$ is assigned to only one element of $B$, so this is a function. The range is $\{b_1, b_3\}$.




The element $a_4 \in A$ is assigned to more than one element of $B$, so this is not a function.

**Example 3.1.3.** For each of the following functions determine:

$f(2), f(0), f(\sqrt{5}), f(\frac{a}{2}), f(b+I)$ and $f(x^2)$.

1. $f(x) = x^2 + 1$.

   $f(2) = 5, f(0) = 1, f(\sqrt{5}) = 6, f(\frac{a}{2}) = \frac{a^2}{4} + 1, f(b+I) = (b+I)^2 + 1, f(x^2) = x^4 + 1.$

2. $f(x) = 2x - 3$.

3. $f(x) = -x + 5$.

Domain, Codomain, Range

**Example 3.1.4.** Determine the domain and the range of each of the following functions:

**1.** $f(x) = \frac{2}{x-1}$.

domain of $f = \{x : x \in \mathbb{R}, x \neq 1\} = \mathbb{R}\setminus\{1\}$,

range of $f = \{x : x \in \mathbb{R}, x \neq 0\} = \mathbb{R}\setminus\{0\}$.

**2.** $f(x) = \frac{-1}{x^2-9}$.

domain of $f = \{x : x \in \mathbb{R}, x \neq 1\} = \mathbb{R}\setminus\{3, -3\}$,

range of $f = \{x : x \in \mathbb{R}, x \neq 0\} = \mathbb{R}\setminus\{0\}$.

**3.** $f(x) = \frac{-x}{x^2+2x+1}$.

domain of $f = \{x : x \in \mathbb{R}, x \neq 1\} = \mathbb{R}\setminus\{-1\}$.

The range of $f$ is $\left\{y \in \mathbb{R} : y \geq \frac{-1}{4}\right\} = [\frac{-1}{4}, \infty)$.

**4.** $f(x) = \sqrt{2x-1}$.

To find the domain of $f$ the value under the root should not be negative. Thus

$$2x - 1 \geq 0 \Longrightarrow 2x \geq 1 \Longrightarrow x \geq \frac{1}{2}$$

hence

domain of $f = \left\{x \in \mathbb{R} : x \geq \frac{1}{2}\right\} = [\frac{1}{2}, \infty)$.

range of $f = \{x : x \in \mathbb{R}, x \geq 0\} = [0, \infty)$.

**5.** $f(x) = \sqrt{1-x}$.

domain of $f = \{x \in \mathbb{R} : x \leq 1\} = (-\infty, 1]$.

range of $f = \{x : x \in \mathbb{R}, x \geq 0\} = [0, \infty)$.

(From the graph or by substituting values for $x$, from the domain, to find $y$)

**6.** $f(x) = \sqrt{x+5}$.

domain of $f = \{x \in \mathbb{R} : x \geq -5\} = [-5, \infty)$.

range of $f = \{x : x \in \mathbb{R}, x \geq 0\} = [0, \infty)$.

(Finding the range is from the graph or by substituting values for $x$, from the domain, to find $y$)

**7.** $f(x) = \frac{\sqrt{x}}{x-1}$.

domain of $f = \{x \in \mathbb{R} : x \geq 0, x \neq 1\} = [0, \infty)\setminus\{1\}$.

range of $f = \{x : x \in \mathbb{R}, x > 0\} = (-1, \infty)$.

(Finding the range is from the graph or by substituting values for $x$, from the domain, to find $y$)

For example:

$x = 0 \Longrightarrow y = 0$

$x = 0.1 \Longrightarrow y = -0.35$

$x = 0.9 \Longrightarrow y = -9.4$

$x = 1.01 \Longrightarrow y = 100.4$

$x = 1.1 \Longrightarrow y = 10.4$

$x = 2 \Longrightarrow y = 1.4$

$x = 2.45 \Longrightarrow y = 1.07$

$x = 10 \Longrightarrow y = 0.3$

$x = 100 \Longrightarrow y = 0.1$

**8.** $f(x) = \frac{\sqrt{4-x}}{x^2-3x-4}$.

domain of $f = \{x \in \mathbb{R} : x \leq 4, x \neq -1, x \neq 4\} = (-\infty, 4] \setminus \{-1, 4\}$.

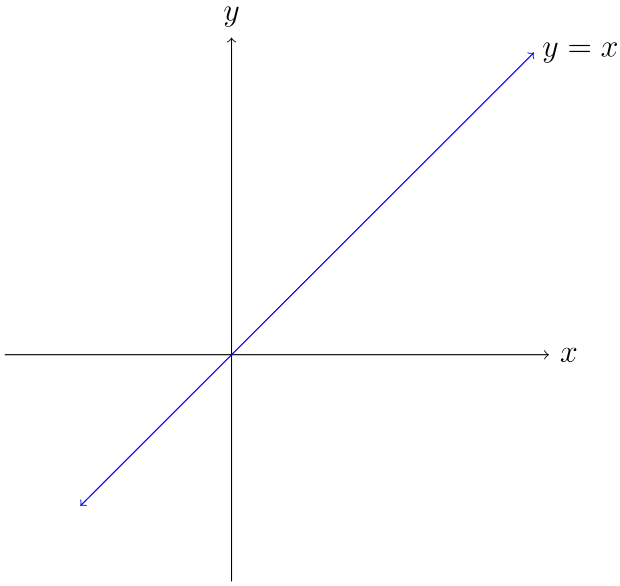
## 3.2 Graphs and Linear Functions

**Definition 3.2.1.** The **graph** of a function $f$ is the set of all points having the coordinates $(x, f(x))$ for $x$ in the domain of $f$. That is if $f : A \to B$ is a function then,

$$G = \{(x, y) \in A \times B : y = f(x)\}.$$

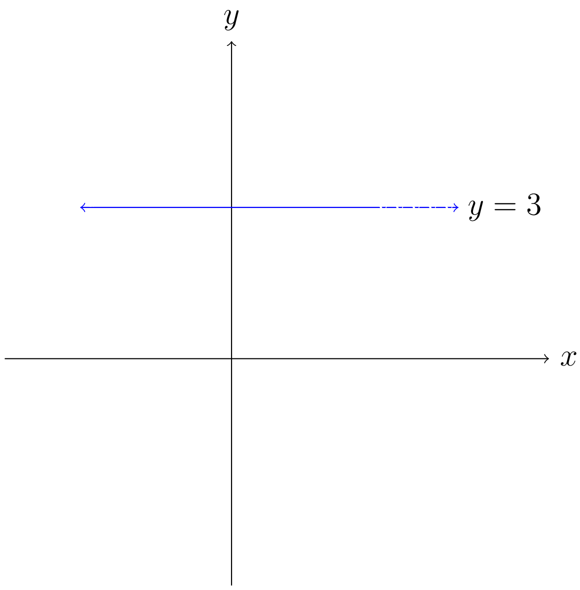
**Example 3.2.2.** Draw the graph of the function $f(x) = x$ (This function is called the **identity function**).

$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\} = \{(0, 0), (1, 1), (2, 2), (-1, -1), (-2, -2), \cdots\}$.



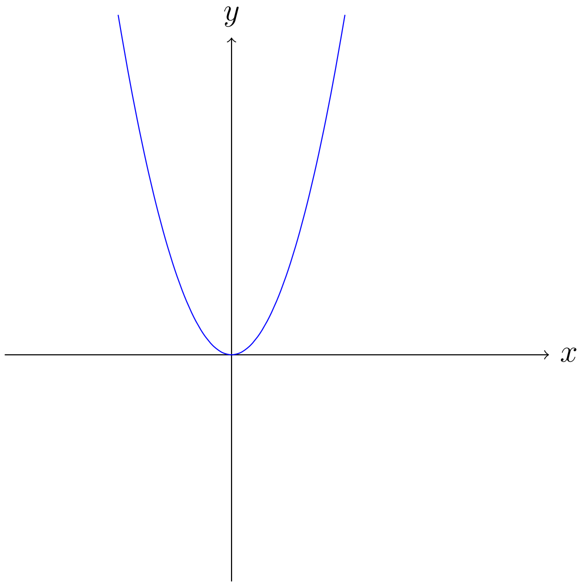

**Example 3.2.3.** Draw the graph of the function $f(x) = 3$ (The function $f(x) = c$, where $c \in \mathbb{R}$ is fixed , called the **identity function**).

$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 3\} = \{(0, 3), (1, 3), (2, 3), (-1, 3), (-2, 3), \cdots\}$.




**Example 3.2.4.** Draw the graph of the function $f(x) = x^2$

$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} = \{(0, 0), (1, 1), (2, 4), (-1, 1), (-2, 4), \cdots\}$.




Linear Functions

**Definition 3.2.5.** A function $f$ is called a linear function if it has the form

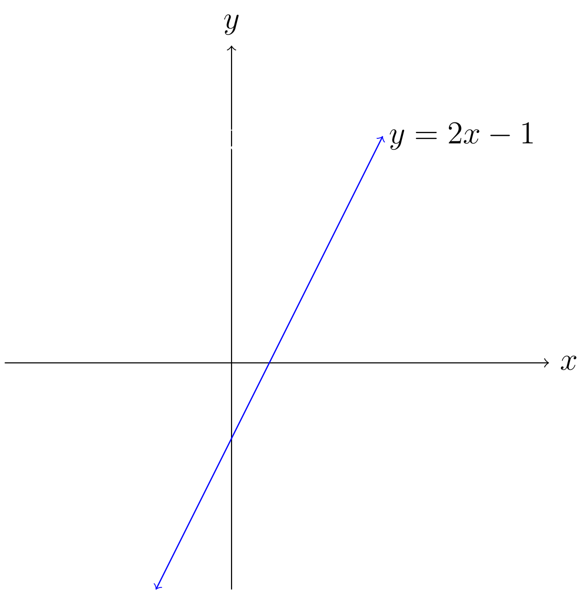
$$f(x) = mx + b$$

for some pair of real numbers $m$ and $b$

**Remark 3.2.6.** The graph of a function is a straight line if and only if the function is linear

**Example 3.2.7.** Draw the graph of the function $f(x) = 2x - 1$

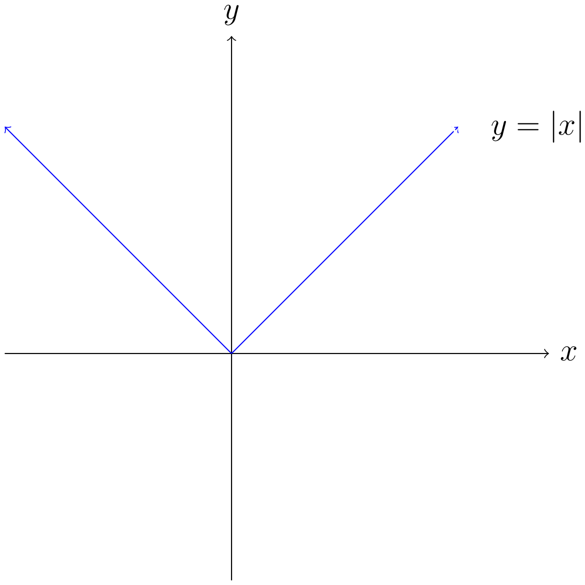
$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 2x - 1\} = \{(0, -1), (1, 1), (2, 3), (-1, -3), (-2, -5), \cdots\}$.



**Example 3.2.8.** Draw the graph of the function $f(x) = |x|$, $x \in \mathbb{R}$. Recall that

$$|x| := \begin{cases} x, & \text{when } x \geq 0, \\ -x & \text{when } x < 0. \end{cases}$$

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = |x|\} = \{(0,0), (1,1), (2,2), (3,3), (-1,-1), (-2,2), (-3,3) \cdots\}.$$

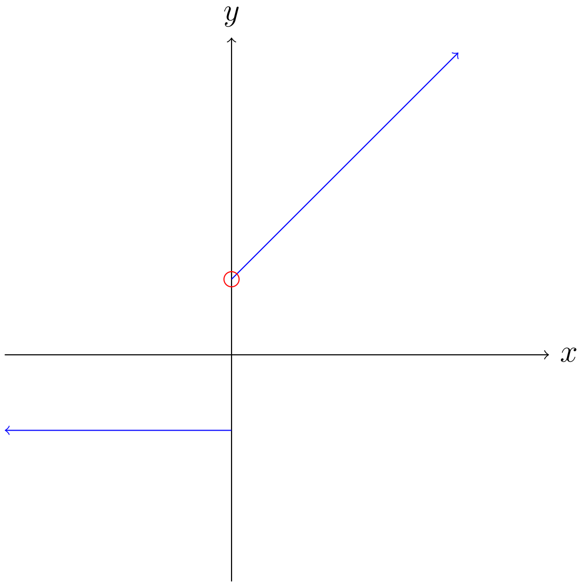



The domain of $f$ is $\mathbb{R}$ and the range of $f$ is $\mathbb{R}^+$.

**Example 3.2.9.** Draw the graph of the the following function

$$f(x) := \begin{cases} -1, & \text{for } x \leq 0, \\ x + 1 & \text{for } x > 0. \end{cases}$$

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = |x|\} = \{(0,0), (1,1), (2,2), (3,3), (-1,-1), (-2,2), (-3,3) \cdots\}.$$



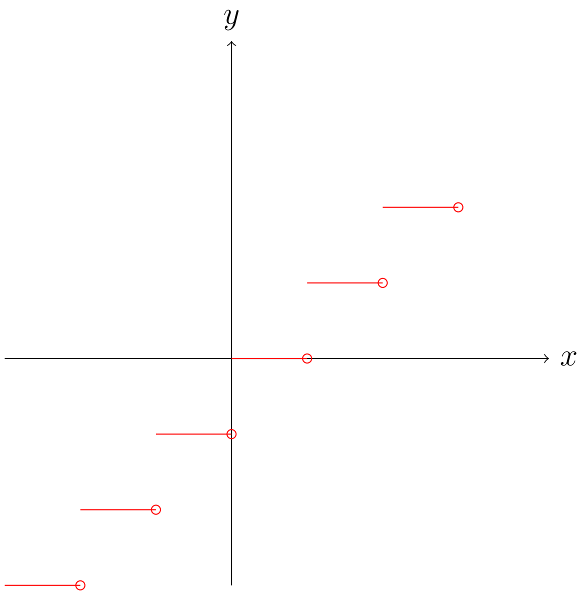

The domain of $f$ is $\mathbb{R}$ and the range of $f = \{y \in \mathbb{R} : y > 1 \text{ and } y = -1\} = (1, \infty) \cup \{-1\}$.

**Example 3.2.10.** Draw the graph of function $f(x) = [x]$, for all $x \in \mathbb{R}$.

If $x$ is any real number, then $[x]$is often used to denote the greatest integer not acceding $x$. In other words, if $x$ is an integer then $[x] = x$, otherwise, $x$ "rounded down" to the next integer is $[x]$. For example: $[5] = 5, [1.2] = 1, [\pi] = 3$ and $[-17.2] = -18$.

$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = [x]\}$

$$= \{(0,0), (0.1,0), (0.9,0)(1,1), (1.1,1), (1.9,1), (2,2), (-0.1,-1), (-1,-1), \cdots\}.$$




The domain of $f$ is $\mathbb{R}$ and the range of $f = \{y \in \mathbb{R} : y > 1 \text{ and } y = -1\} = (1, \infty) \cup \{-1\}$.

**Example 3.2.11.** Draw the graph of each of the following function and determine the domain and the range.

1. $f(x) = |x| + 1$.
2. $f(x) = |x| - 1$.
3. $f(x) = |2x|$.
4. $f(x) = [x] - 1$.
5. $f(x) = [2x]$.
6. $f(x) = [x] + 1$.
7.
$$f(x) := \begin{cases} -2, & \text{for} \quad x \leq 1, \\ x & \text{for} \quad x > 1. \end{cases}$$
8.
$$f(x) := \begin{cases} x, & \text{for} \quad x < 0, \\ 2x & \text{for} \quad x \geq 0. \end{cases}$$
9. $f(x) = \frac{x^2}{4} + 3$.
10. $f(x) = x^2 - 8x + 16$.
11. $f(x) = -2x^2 - 6x$.
12.
$$f(x) := \begin{cases} x + 1, & \text{for} \quad x \leq 0, \\ -x + 1 & \text{for} \quad x \geq 0. \end{cases}$$

Solution: Homework

## 3.3 Inverse of Functions

**Definition 3.3.1** (Functions from other functions)**.**

Let $A, B$ be a non-empty sets. Suppose that $f : A \to B$ and $g : A \to B$, then

1. The pointwise sum of $f$ and $g$ denoted by $f + g$, is a function from $A$ to $B$, defined by $(f + g)(a) = f(a) + g(a)$, for all $a \in A$.
2. $f - g$, is a function from $A$ to $B$, defined by $(f - g)(a) = f(a) - g(a)$, for all $a \in A$.
3. The pointwise product of $f$ and $g$ denoted by $fg$, is a function from $A$ to $B$, defined by $(fg)(a) =$

$f(a)g(a)$, for all $a \in A$.

4. $\frac{f}{g}$, is a function from $A$ to $B$, defined by $(\frac{f}{g})(a) = \frac{f(a)}{g(a)}$, for all $a \in A$ such that $g(a) \neq 0$.

**Remark 3.3.2.** The domain of $f + g$ is the set of all real Numbers that are in the domain of $f$ **AND** in the domain of $g$. That is

$$\text{domain}(f+g) = \{x \in \mathbb{R} : x \in \text{ domain } f \cap \text{domain } g\}.$$

The same rule applies when we subtract, multiply or divide, except divide has one extra rule, which is the denominator should not be zero.

**Example 3.3.3.** Let $f(x) = 2x + 3$ and $g(x) = x^2$, for all $x \in \mathbb{R}$. Then

1. $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 3$.
2. $(f - g)(x) = f(x) + g(x) = 2x + 3 - x^2$.
3. $(fg)(x) = f(x)g(x) = (2x + 3)x^2 = 2x^3 + 3x^2$.
4. $(f/g)(x) = f(x)/g(x) = \frac{2x+3}{x^2}$, $x \neq 0$.

**Definition 3.3.4.** Let $f : A \to B$ and $g : A_1 \to B_1$ be functions. Then we say that $f$ is equal $g$ if and only if

1. $A = A_1$
2. $B = B_1$
3. $f(x) = g(x)$ for all $x \in A$.

Then we write $f = g$

**Example 3.3.5.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = |x|$. Suppose that $g : \mathbb{Z} \to \mathbb{N}$ be a function such that $f(x) = |x|$. Then $f \neq g$.*

**Definition 3.3.6** (Injective Function)**.**

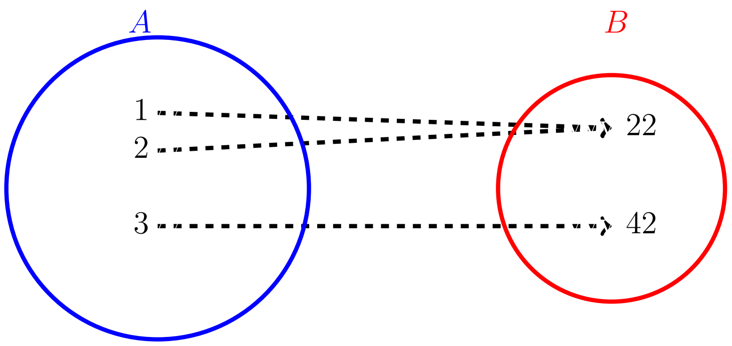
Let $A$ and $B$ be a non-empty sets. A function $f : A \to B$ is **injective** or **one-to-one** if and only if:

$$\boxed{x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2) \quad \text{for all } x_1 \text{ and } x_2 \quad \text{in the domain of f,} \qquad (3.1)}$$

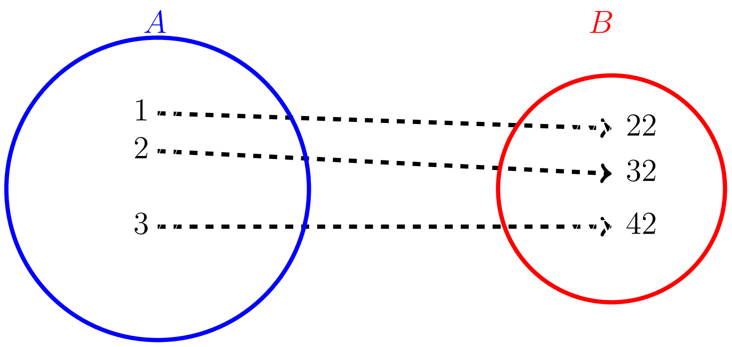
or

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \quad \text{for all } x_1 \text{ and } x_2 \text{ in the domain of } f. \tag{3.2}$$

**Example 3.3.7.** Consider the following functions:




This function is not injective, since $1 \neq 2$ but $f(1) = f(2) = 22$.




This function is injective.

**Example 3.3.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 2x + 5$. Verify that $f$ is injective.

First Solution

Let $x_1, x_2 \in \mathbb{R}$. Suppose that

$x_1 \neq x_2$ this implies $\Rightarrow 2x_1 \neq 2x_2$ this implies $\Rightarrow 2x_1 + 5 \neq 2x_2 + 5$,

thus $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

Second Solution

Let $x_1, x_2 \in \mathbb{R}$. Suppose that

$f(x_1) = f(x_2)$ this implies $\Rightarrow 2x_1 + 5 = 2x_2 + 5$

this implies $\Rightarrow 2x_1 = 2x_2$,

thus $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

**Example 3.3.9.** Verify that the following functions are injective of not.

1. $f(x) = x^3$, for all $x \in \mathbb{R}$.
2. $g(x) = 3x^2 + 1$, for all $x \in \mathbb{R}$.

Solution 2. Let $x_1, x_2 \in \mathbb{R}$. Suppose that $f(x_1) = f(x_2)$, this implies $\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$. Thus $f$ is injective.

Solution 1. If $x_1 = 1$ and $x_2 = -1$, then $g(x_1) = 4 = g(x_2)$, that is there exists $x_1 \neq x_2$ and $g(x_1) = g(x_2)$. Hence $g$ is not injective.

**Example 3.3.10.** Decide in each case whether the following functions is one-to-one. Justify your answer.

1. $f(x) = 10x$
2. $f(x) = -7$
3. $f(x) = 7x + 1$
4. $g(x) = 0$
5. $f(x) = -5x + 7$
6. $g(x) = 2 - x^2$
7. $h(x) = 3x^2 \quad \text{for} \quad x \geq 0$
8. $h(x) = |x|$
9. $h(x) = 1 + \sqrt{x} \quad \text{for} \quad x \geq 0$.

Solution

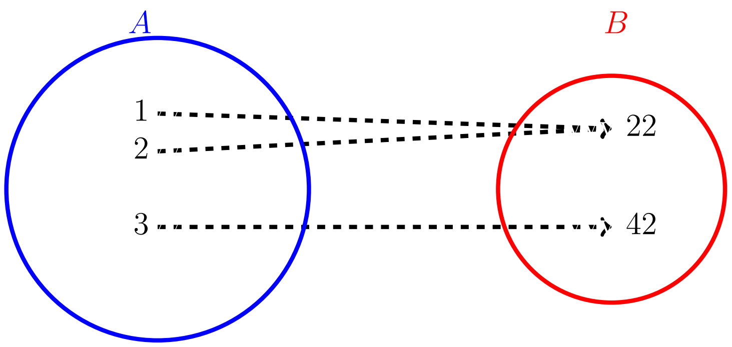
Homework

**Definition 3.3.11** (Surjective function)**.**

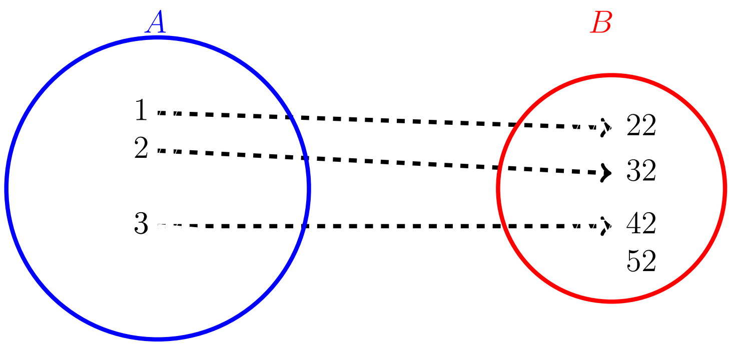
Let $A$ and $B$ be a non-empty sets. A function $f : A \to B$ is **surjective** or **onto** if and only if **range of** $f = B$. In other words,

$$\textbf{For all } y \in B, \textbf{there exists } x \in A, \textbf{such that } y = f(x).$$

**Example 3.3.12.** Consider the following functions:




This function is surjective.




This function is not surjective, because there exists $y = 52 \in B$ and it does not exist $x \in A$ such the $f(x) = y$.

**Example 3.3.13.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 3x - 7$. Show that $f$ is surjective (onto).

Solution:

Let $y \in \mathbb{R}$. Set $y = f(x) \Rightarrow y = 3x - 7$.

Solve for $x$: $y = 3x - 7 \Rightarrow 3x = y + 7 \Rightarrow x = \frac{y+7}{3}$

Now, $f(x) = 3\left(\frac{y+7}{3}\right) - 7 = y$.

Thus for each $y \in \mathbb{R}$ (codomail of $f$) there exists $x \in \mathbb{R}$ (domain of $f$) such that $f(x) = y$. Hence $f$ is surjective.

**Example 3.3.14.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x^2 + 1$. Verify that if $f$ is surjective or not.

Solution: This function is not surjective, because there exists $y = -1 \in B = \mathbb{R}$ and it does not exist $x \in A = \mathbb{R}$ such that $f(x) = y$.

**Definition 3.3.15** (Bijective Function)**.**

Let $A, B$ be a non-empty sets. A function $f : A \to B$ is called **bijective** if and only if $f$ is **injective (one-to-one)** and it is **surjective (onto)**.

If $f : A \to B$ is bijective, then:

- For all $x \in A$ there exists only one image $f(x) \in B$.
- For all $y \in B$ there exists only one element $x \in A$ such that $f(x) = y$ thus $f$ is some time called **one to one correspondence**.

**Example 3.3.16.** Let $A = \{1, 3, 5, 7, 9, \cdots\}$ and $B = \{2, 4, 6, 8, 10, \cdots\}$.

Suppose that $f : A \to B$ defined by $f(x) = 2x$ and $g : A \to B$ defined by $g(x) = x + 1$. Verify that if $f$ and $g$ are bijective or not.

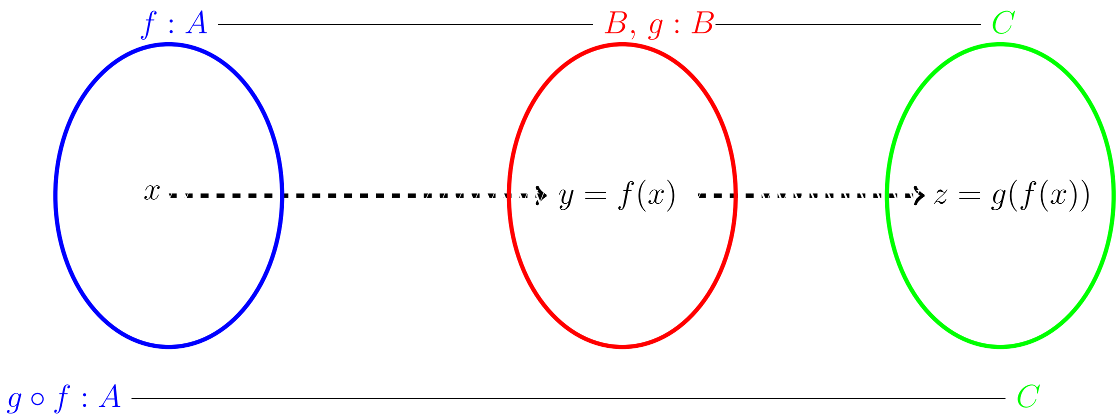
Solution

**Example 3.3.17.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 2x^3 - 7$. Verify that if $f$ is bijective or not.

Solution

**Definition 3.3.18** (Composite Function)**.**

Let $A, B$ and $C$ be non-empty sets. Suppose that $f : A \to B$ and $g : A \to C$ are functions.




Then $(g \circ f) : A \to C$ is a function, called the **composite function** defined by

$$(g \circ f)(x) = g(f(x)).$$

**Example 3.3.19.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(x) = 4x^2$, for all $x \in \mathbb{N}$. Suppose that $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(x) = 3x + 1$, for all $x \in \mathbb{N}$. Then

$(g \circ f) : \mathbb{N} \to \mathbb{N}$ is a function and

$(g \circ f)(x) = g(f(x)) = g(4x^2) = 3(4x^2) + 1 = 12x^2 + 1.$

**Theorem 3.3.20.** *Let $A, B$ and $C$ be non-empty sets. Suppose that $f : A \to B$ and $g : B \to C$ are functions. Then*

***1.*** *If $f$ and $g$ are injective, then $(g \circ f)$ is injective.*

***2.*** *If $f$ and $g$ are surjective, then $(g \circ f)$ is surjective.*

***3.*** *If $f$ and $g$ are bijective, then $(g \circ f)$ is bijective.*

*Proof.* 1. Let $x_1, x_2 \in A$. Suppose that

$$(g \circ f)(x_1) = (g \circ f)(x_2),$$

this implies

$$g(f(x_1)) = g(f(x_2)),$$

since $g$ is injective then

$$f(x_1) = f(x_2)$$

since $f$ is injective then $x_1 = x_2$.

*Proof* 2. Let $z \in C$, to prove $g \circ f$ is surjective we have to find $x \in A$ such that $(g \circ f)(x) = z$.

Since $g$ is onto then there exists $y \in B$ such that $g(y) = z$.

Since $f$ is onto then there exists $x \in B$ such that $f(x) = y$.

This means that for every $z \in C$ then there exists $x \in A$ such that $g(f(x)) = z$, that is $(g \circ f)(x) = z$.

Thus $g \circ f$ is onto.

*Proof* 3. Homework

□

**Theorem 3.3.21.** *Let $A, B$ and $C$ be non-empty sets. Suppose that $f : A \to B$ and $g : B \to C$ are functions. Then*

***1.*** *If $(g \circ f)$ is injective then $f$ is injective.*

***2.*** *If $(g \circ f)$ is surjective then $g$ is surjective.*

*Proof.* 1. Let $x_1, x_2 \in A$. Suppose that

$$f(x_1) = f(x_2)$$

then

$$g(f(x_1)) = g(f(x_2))$$

using the definition of composite function we get

$$(g \circ f)(x_1) = (g \circ f)(x_2)$$

Since $g \circ f$ is one to one, then $x_1 = x_2$.

Hence $f$ is one-to-one.

*Proof* 2. Let $z \in C$.

Since $g \circ f$ is onto then there exists $x \in A$ such that $(g \circ f)(x) = z$.

That is $g(f(x)) = z$.

Since $f(x) \in B$, then $g$ is onto.

□

*Inverse of a function*

**Definition 3.3.22.** Let $f : A \to B$ be a function. Then $f$ has **an inverse** if there exists a function $g : B \to A$ such that

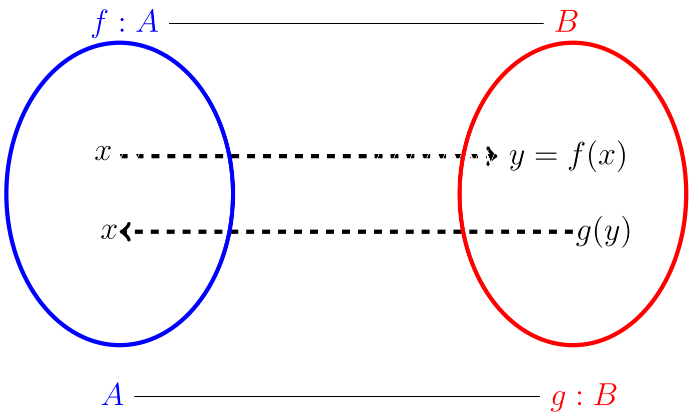
$$(g \circ f)(x) = x \qquad \text{for all } x \in A$$

and

$$(f \circ g)(x) = x \qquad \text{for all } x \in B$$

A function $f$ has inverse if and only if it is bijective (one-to-one and onto).

**We will use the symbol $f^{-1}$ to represent the inverse of $f$.**




$$y = f(x) \Longleftrightarrow f^{-1}(y) = x.$$

To finding $F^{-1}(x)$, given $f(x)$

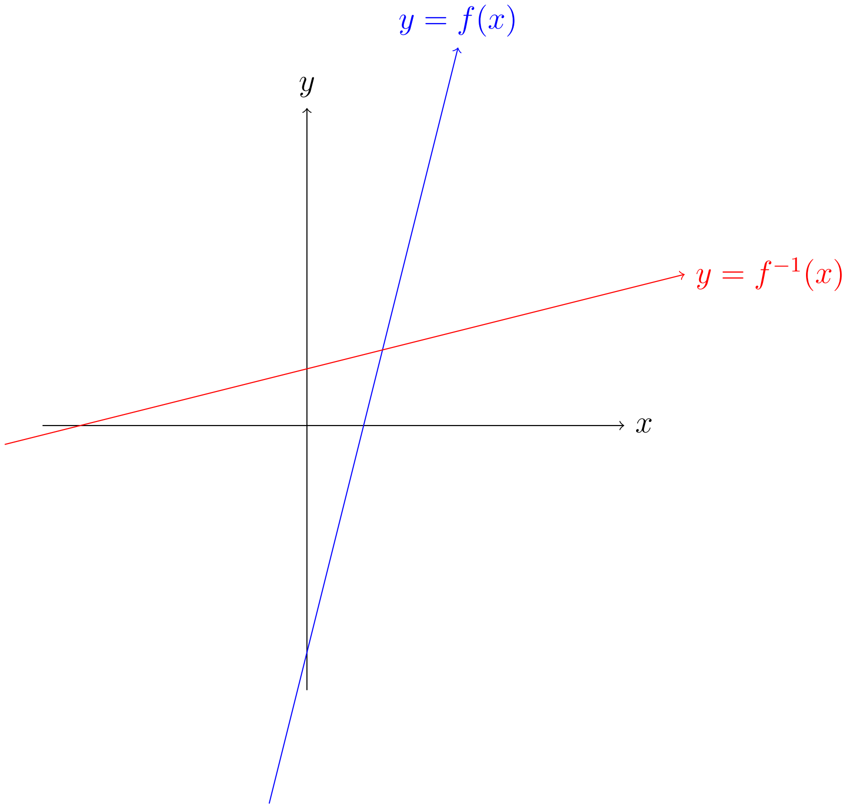
1. Set $y = f(x)$.
2. Solve for $x$ to get $x = f^{-1}(y)$.
3. Replace $y$ by $x$ to get $f^{-1}(x)$.

**Example 3.3.23.** Let $f(x) = 4x - 3$, find $f^{-1}(x)$. Draw the graph of $f$ and $f^{-1}$.

<u>Solution:</u> Let $y = f(x)$.

$y = 4x - 3 \Rightarrow x = \frac{y+3}{4}$.

Thus $f^{-1}(x) = \frac{x+3}{4}$.




**Example 3.3.24.** Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) = x^2 + 5$. Find $f^{-1}(x)$, draw the graph of $f$ and $f^{-1}$.

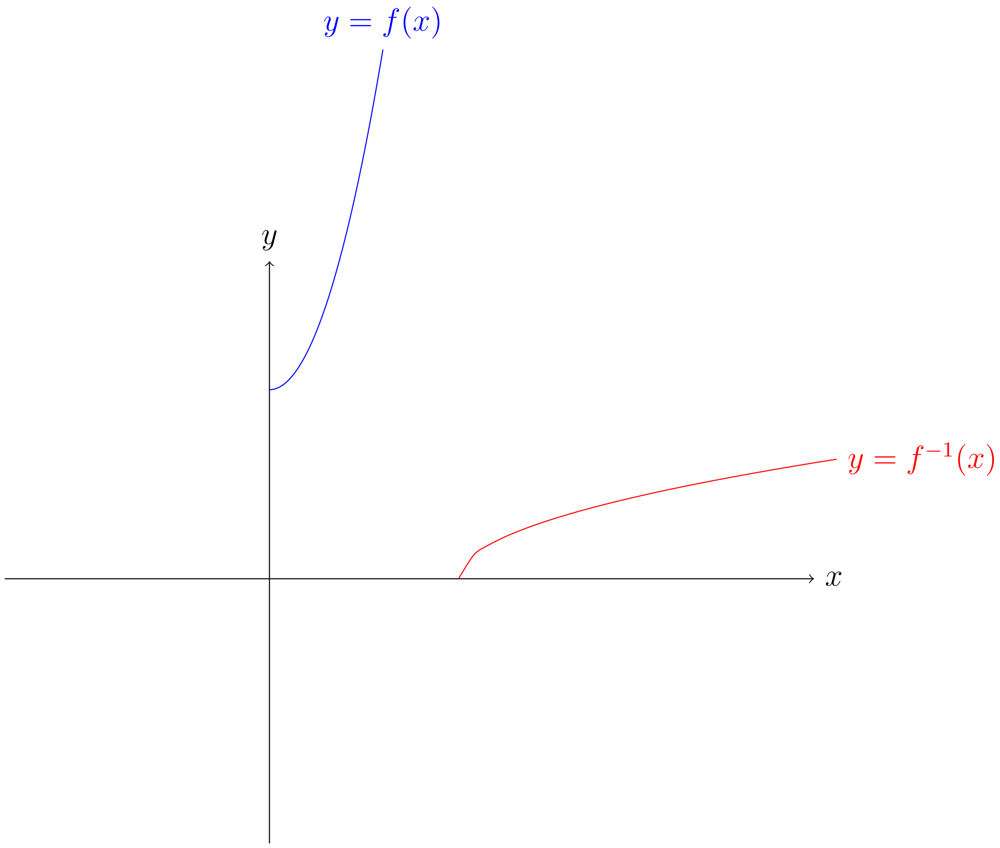
Solution: Let $y = f(x)$.

$y = x^2 + 5 \Rightarrow x^2 = y - 5 \Rightarrow x = \sqrt{y - 5}$.

Thus $f^{-1}(x) = \sqrt{x - 5}$ for $x \geq 5$.

Note that the domain of $f$ is $[0, \infty)$ and range of $f$ is $[5, \infty)$.

The domain of $f^{-1}$ is $[5, \infty)$ and range of $f^{-1}$ is $[0, \infty)$.




**Example 3.3.25.** Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(x)$, draw the graph of $f$ and $f^{-1}$.

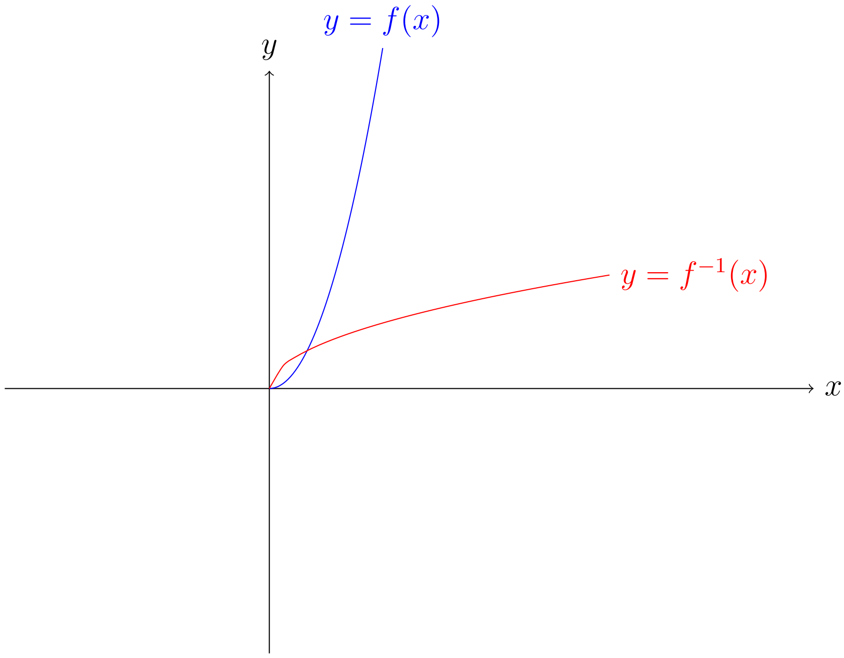
<u>Solution:</u> Let $y = f(x)$.

$y = x^2 \Rightarrow x = \sqrt{y}$.

Thus $f^{-1}(x) = \sqrt{x}$ for $x \geq 0$.

Note that the domain of $f$ is $[0, \infty]$ and range of $f$ is $[0, \infty)$.

The domain of $f^{-1}$ is $[0, \infty]$ and range of $f^{-1}$ is $[0, \infty)$.




**Example 3.3.26.** For each of the following functions:

(a) determine $f^{-1}$.

(b) Verify that $(f^{-1} \circ f)(x) = x$ for each $x$ in the domain of $f$ and $(f \circ f^{-1})(x) = x$ for each $x$ in the domain of $f^{-1}$.

(c) Draw the graph of $f$ and $f^{-1}$.

1. $f(x) = 2x$
2. $f(x) = 4x$
3. $f(x) = -3x$
4. $f(x) = x + 1$
5. $f(x) = 5 - x$
6. $f(x) = x - 3$
7. $f(x) = 3 - 2x$
8. $f(x) = 3x + 4$
9. $f(x) = 6 - x$
10. $f(x) = \sqrt{x} + 1$
11. $f(x) = \sqrt{2x}$
12. $f(x) = 1 - 2\sqrt{x}$
13. $f(x) = 2x^2$
14. $f(x) = x^2 - 1$
15. $f(x) = 1 - x^2$

16. $f(x) = \sqrt{1-x^2}$, for all $0 \le x \le 1$

Solution: Homework

**Example 3.3.27.**

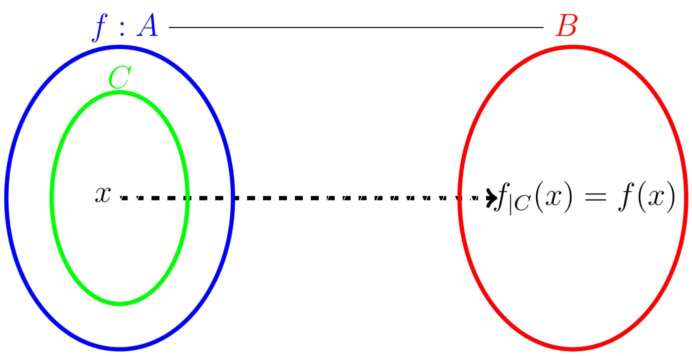
1. Let $f(x) = \sqrt{4x+1}$. Find functions $g$ and $h$ such that $h \circ g = f$.
2. Let $f(x) = x^3 - 1$. Find functions $g$ and $h$ such that $h \circ g = f$.

Solution: Homework

## 3.4 Some types of functions

**Definition 3.4.1** (Restriction of Function)**.**
Let $A, B$ be a non-empty sets and $C \subseteq A$. Suppose that $f : A \to B$ is a function. Then the **restriction of $f$ to $C$** is the function $f_{|C} : C \to B$ such that

$$f_{|C}(x) = f(x), \qquad \text{for all} \;\; x \in \mathbb{C}.$$




**Example 3.4.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = x^2$. Suppose that $g : \mathbb{N} \to \mathbb{R}$ is a function defined as $g(x) = x^2$. Since $\mathbb{N} \subseteq \mathbb{R}$ and $g(x) = f(x)$ for all $x \in \mathbb{N}$, then $g$ is a restriction of $f$ to $\mathbb{N}$, that is $g = f_{|\mathbb{N}}$.

**Lemma 3.4.3.** *Let $A, B$ be a non-empty sets and $C \subseteq A$. Suppose that $f : A \to B$ is a one-to-one function. Prove that $f_{|\mathbb{N}} : C \to B$ is a one-to-one function.*

*Proof.*

$\square$

**Definition 3.4.4** (Extension of Function)**.**
Let $D, B$ be a non-empty sets and $A \subseteq D$. Suppose that $f : A \to B$ is a function. Then the **extension of $f$ to $D$** is the function $g_{|A} : D \to B$ such that

$$g_{|A}(x) = f(x), \quad \text{for all} \;\; x \in \mathbb{A}.$$

**Example 3.4.5.** Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined as $g(x) = x^2$. Suppose that $f : \mathbb{Z} \to \mathbb{R}$ is a function defined as $f(x) = x^2$. Since $\mathbb{Z} \subseteq \mathbb{R}$ and $g(x) = f(x)$ for all $x \in \mathbb{Z}$, then $g$ is an extension of $f$ to $\mathbb{R}$, that is $g = g_{|\mathbb{Z}}$.

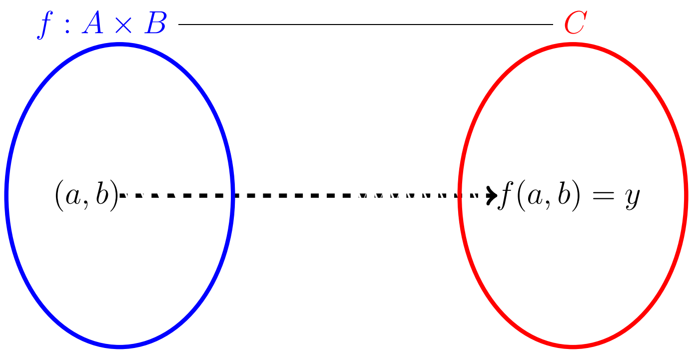
**Definition 3.4.6** (The Characteristic Function)**.**
Let $A$ be a non-empty set and $C \subseteq A$. Suppose that $B = \{0, 1\}$. Then the **characteristic function** $I_A : A \to B$ is defined as:

$$I_A := \begin{cases} 1, & \text{when } x \in A, \\ 0 & \text{when } x \notin A. \end{cases}$$

**Definition 3.4.7** (Function of Several Variable)**.**
Let $A, B$ and $C$ be a non-empty sets. A function $f : A \times B \to C$ is function in two variables. The domain of $f$ is $A \times B$, where

$$A \times B = \{(a, b) : a \in A, \text{ and } b \in B\}$$

and the codomain of $f$ is $B$. The image of $(a, b) \in A \times B$ under the function $f$ is $f(a, b)$ which is an element in $B$.




**Example 3.4.8.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$f(h, r) = \pi r^2 h \quad \text{(The formula of the volume of a cylinder of radius } r \text{and height } h\text{).}$$

Then $f$ is a function of two variables.

We can generalized this concept to functions that have domain and codomain are a Cartesian product of more than two sets. For example:

The function $f : \mathbb{R}^4 \to \mathbb{R}^2$ which is defined as

$$f(x, y, z, w) = (3x + y, 4z + 5w),$$

is a function in four variables. The domain of $f$ is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$.