Analytical Mechanics

Dr. Bassma H. Hamad
Al-Nahrain university

Syllabus

- ***Introduction.***
- ***Lagrange’s Equation.***
- ***Small Oscillations.***
- ***Hamilton’s equations of motion.***

System of Particles

With respect to a system of 3-dimensional coordinates, we need 3n number of independent variables to describe the position of a system consisting of n number of particles.

If there are k number of constraints (restrictions), then we need only 3n-k number of independent variables.

We define ***Degrees of freedom*** *of the system as 3n-k.*

By looking at the system we can define these independent variables and they are called ***generalized coordinates*** *and they are denoted by* $q_1, q_2, \ldots, q_n$.

Lagrange's Equations

Result I

Suppose $\underline{r}_i$ *is the position vector of the* i^{th} *particle and* q_j *is the* j^{th} *generalized coordinate.*

$$\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j} \quad .$$

Proof : Since $\underline{r}_i = \underline{r}_i\left(q_1, q_2, \ldots\ldots\ldots\ldots\ldots\ldots, q_n, t\right)$

$$d\underline{r}_i = \sum_{j=1}^{N} \frac{\partial \underline{r}_i}{\partial q_j} dq_j + \frac{\partial \underline{r}_i}{\partial t} dt \qquad \text{——— Eq. 1}$$

Therefore $$\dot{\underline{r}}_i = \frac{d\,\underline{r}_i}{dt}$$

$$= \frac{1}{dt}\left[\sum_{j=1}^{N} \frac{\partial \underline{r}_i}{\partial q_j}\, dq_j + \frac{\partial \underline{r}_i}{\partial t}\, dt\right]$$

$$= \sum_{j=1}^{N} \frac{\partial \underline{r}_i}{\partial q_j}\, \dot{q}_j + \frac{\partial \underline{r}_i}{\partial t}$$

Hence $$\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j} \quad .$$

Result II

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\underline{r}}_i}{\partial q_j}.$$

Proof :

Since $$\dot{\underline{r}}_i = \sum_{m=1}^{N} \frac{\partial \underline{r}_i}{\partial q_m} \dot{q}_m + \frac{\partial}{\partial t} \underline{r}_i$$

$$\frac{\partial \dot{\underline{r}}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_{m=1}^{N} \frac{\partial \underline{r}_i}{\partial q_m} \dot{q}_m + \frac{\partial \underline{r}_i}{\partial t}\right)$$

$$= \sum_{m=1}^{N} \frac{\partial}{\partial q_j}\left(\frac{\partial \underline{r}_i}{\partial q_m}\right) \dot{q}_m + \frac{\partial}{\partial q_j}\left(\frac{\partial \underline{r}_i}{\partial t}\right)$$

$$= \sum_{m=1}^{N} \frac{\partial}{\partial q_m}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)\dot{q}_m + \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)$$

Hence the result is proved.

We have

$$\boxed{\frac{\partial \underline{\dot{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j}}$$ — **Result 1**

$$\boxed{\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right) = \frac{\partial \underline{\dot{r}}_i}{\partial q_j}}$$ — **Result 2**

Kinetic energy of the system is $T = \sum_{i=1}^{n} \frac{1}{2} m_i \, \underline{\dot{r}}_i \cdot \underline{\dot{r}}_i$

$$\therefore \frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j} \sum_{i=1}^{N} \frac{1}{2} m_i \, \underline{\dot{r}}_i \cdot \underline{\dot{r}}_i$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial}{\partial q_j} \left(\underline{\dot{r}}_i \cdot \underline{\dot{r}}_i \right)$$

$$\therefore \frac{\partial T}{\partial q_j} = \sum_{i=1}^{N} m_i \, \underline{\dot{r}} \cdot \frac{\partial \underline{\dot{r}}_i}{\partial q_j} \, .$$

Also

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \sum_{i=1}^{N} \frac{1}{2} m_i \, \underline{\dot{r}}_i \cdot \underline{\dot{r}}_i$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial}{\partial \dot{q}_j} \left(\underline{\dot{r}}_i \cdot \underline{\dot{r}}_i \right)$$

$$= \sum_{i=1}^{N} m_i \, \underline{\dot{r}}_i \cdot \frac{\partial \underline{\dot{r}}_i}{\partial \dot{q}_j}$$

$$= \sum_{i=1}^{N} m_i \, \underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

By ***Result 1***

Hence

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{n} m_i \, \underline{\dot{r}}_i \cdot \frac{\partial \underline{\dot{r}}_i}{\partial q_j}$$

Result 3

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i \, \underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

Result 4

Suppose the system of particles changed slightly without changing the time t then ***Eq. 1*** *becomes*

$$d\underline{r}_i = \sum_{j=1}^{N} \frac{\partial \underline{r}_i}{\partial q_j} dq_j$$

Work done

$$dw = \sum_{j=1}^{N} \underline{F}_i . d\underline{r}_i$$

Here $\underline{F}_i$ *is the external force on the* i^{th} *particle.*

$$= \sum_{i=1}^{N} \underline{F}_i \cdot \sum_{j=1}^{N} \frac{\partial \underline{r}_i}{\partial q_j} \quad dq_j$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \quad dq_j$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \quad dq_j$$

$$= \sum_{j=1}^{N} Q_j \, dq_j$$

Here $$Q_j = \sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

is called ***generalized force*** *associated with generalized coordinate* q_j *.*

Also $w = w(q_1, q_2, \ldots\ldots\ldots\ldots q_N)$ *gives us*

$$dw = \sum_{j=1}^{N} \frac{\partial w}{\partial q_j} dq_j$$

$$\therefore \sum_{j=1}^{N} \frac{\partial w}{\partial q_j} dq_j = \sum_{j=1}^{N} Q_j \; dq_j$$

$$\Rightarrow \sum_{j=1}^{N} \left(\frac{\partial w}{\partial q_j} - Q_j \right) dq_j = o$$

Since dq_j *are all independent above equation yields*

$$Q_j = \frac{\partial w}{\partial q_j}$$ ——— ***Eq. 2***

Applying Newton's 2nd Law $\underline{F}_i = m_i \ddot{\underline{r}}_i$ *to the i*th *particle we have*

$$\therefore \; \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} = m_i \ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

$$= m_i \frac{d}{dt}\underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

$$= m_i \left[\frac{d}{dt}\left(\underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) - \underline{\dot{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j} \right) \right] \qquad \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j} \right) = \frac{\partial \underline{\dot{r}}_i}{\partial q_j}$$

$$= m_i \left[\frac{d}{dt}\left(\underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) - \underline{\dot{r}}_i \cdot \left(\frac{\partial \underline{\dot{r}}_i}{\partial q_j} \right) \right]$$

By ***Result 2***

$$\therefore\ Q_j = \sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

$$= \sum_{i=1}^{n} m_i \left[\frac{d}{dt}\left(\underline{\dot{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) - \underline{\dot{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \underline{\dot{r}}_i}{\partial q_j} \right) \right]$$

$$= \frac{d}{dt}\sum_{i=1}^{n} m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} - \sum_{i=1}^{n} m_i \dot{\underline{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \dot{\underline{r}}_i}{\partial q_j}\right)$$

When ***Result 3*** *and* ***Result 4*** *are applied in above equation, it reads as*

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_j}$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

This is called Lagrange's equation of motion.

Special case

Suppose all the forces are conservative. i.e. there exists a scalar function $V = V(q_1, q_2, \ldots\ldots\ldots\ldots q_N, t)$ *called potential function.*

By the definition of the potential function $\dfrac{\partial V}{\partial \dot{q}_j} = 0.$

Definition

Lagrangian *or* ***Lagrange's Function L*** *of the system is defined as the difference of Kinetic energy and the Potential energy and denoted by* ***L****.*

i.e. $\boldsymbol{L = T - V}$

Here $dw = \underline{F}.d\underline{r}$

$$= -\nabla V.d\underline{r}$$

$$= -dV$$

$$\therefore \frac{\partial w}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Hence by ***Eq. 2*** $Q_j = \frac{\partial w}{\partial q_j}$ *we get*

$$-\frac{\partial V}{\partial q_j} = Q_j$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\right) = 0 \qquad Since\ \frac{\partial V}{\partial \dot{q}_j} = 0.$$

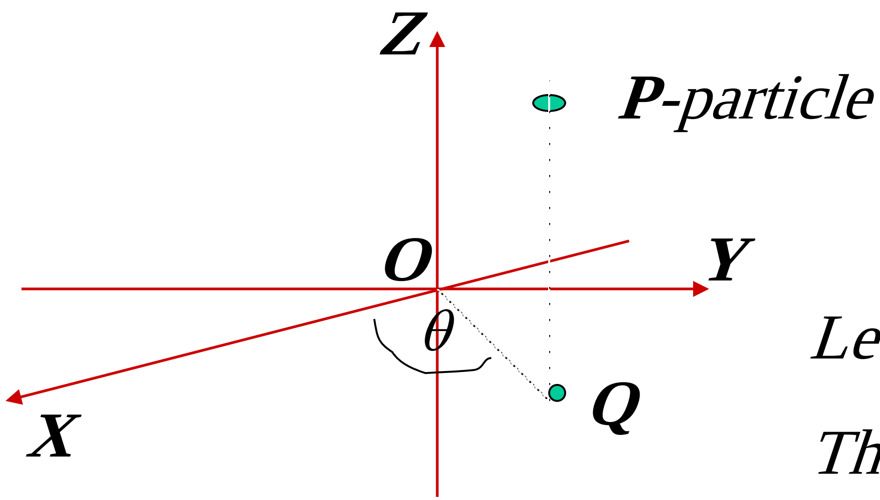
$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T - V)\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

This is the Lagrange's equation for a conservative system

E.g. ***(Question No. 1 of Exercises)***

A particle of mass m moves in a conservative force field. Find the Lagrangian and equations of motions in cylindrical polar coordinates.



Let OQ= r , P $\equiv (x, y, z)$

Then $\underline{r} = \overrightarrow{OP}$

$$= r\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j} + z\underline{k}\right)$$

$$\therefore \dot{\underline{r}} = \left(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\right)\underline{i} + \left(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\right)\underline{j} + \dot{z}\underline{k}$$

$$\therefore \dot{\underline{r}}.\dot{\underline{r}} = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$$

$$\therefore L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r,\theta,z)$$

$$\therefore L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r, \theta, z)$$

Therefore

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta} \qquad \frac{\partial L}{\partial z} = \frac{\partial V}{\partial z}$$

Hence equations of motions are

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = o.$$

$$m\frac{d}{dt}\left(r^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} = o.$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = o.$$

E.g. ***(Question No. 2 of Exercises)***

Suppose that the particle, in the previous example moves in the Oxy plane and V=V(r) only. If at time t=0 the particle on the Ox axis of distance a and the velocity of the particle is v_o in the direction of the positive Oy axis. Find the velocity of the particle.

Solution :

The equations $m\ddot{r} - mr\dot{\theta}^2 + \dfrac{\partial V}{\partial r} = o.$ $m\dfrac{d}{dt}\left(r^2\dot{\theta}\right) + \dfrac{\partial V}{\partial \theta} = o.$

$$m\ddot{z} + \frac{\partial V}{\partial z} = o.$$ *become*

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{dV}{dr} = o. \qquad m\frac{d}{dt}\left(r^2\dot{\theta}\right) = o$$

$$\Rightarrow \quad r^2\dot{\theta} = C$$

$$\Rightarrow \quad C = r^2\dot{\theta} = r\, r\dot{\theta}$$

$$\Rightarrow \quad C = av_o \;\Rightarrow \quad r^2\dot{\theta} = av_o$$

$$\therefore \left(r^2\dot{\theta}\right)^2 = a^2 v_o^2 \;\Rightarrow r\dot{\theta}^{\,2} = \frac{a^2 v_o^2}{r^3}.$$

So the equation

$$m\ddot{r} - mr^{-3}(av)^2 + \frac{dV}{dr} = o.$$

becomes

$$m\ddot{r} - ma^2 v_o^2 \frac{1}{r^3} + \frac{d}{dr}V(r) = o$$

Further $\quad \ddot{r} = \frac{d}{dt}\dot{r}$

$$= \frac{d\dot{r}}{dr}\,\frac{dr}{dt}$$

$$= \dot{r}\;\frac{d\dot{r}}{dr}$$

Hence $\dot{r}\frac{d\dot{r}}{dr} - a^2 v_o^2 \frac{1}{r^3} = -\frac{1}{m}\frac{d}{dr}V(r).$

$$\int \dot{r}\, d\dot{r} = \int \left(a^2 v_o^2 \frac{1}{r^3} - \frac{1}{m}\frac{d}{dr}V(r) \right) dr$$

$$\frac{1}{2}\dot{r}^2 = -\frac{1}{2}a^2 v_o^2 \frac{1}{r^2} - \frac{1}{m}V(r) + C$$

$$o = -\frac{1}{2}a^2 v_o^2 \frac{1}{a^2} - \frac{1}{m}V(a) + C$$

$$\Rightarrow \quad C = \frac{1}{2}v_o^2 + \frac{1}{m}V(a)$$

$$\Rightarrow \quad v^2 = -a^2 v_o^2 \frac{1}{r^2} - 2\frac{1}{m}V(r) + v_o^2 - \frac{2}{m}V(a)$$

$$i.e. \quad v^2 = \frac{v_o^2}{r^2}\left(r^2 - a^2\right) + \frac{2}{m}\left(V(a) - V(r)\right).$$

Now we have velocity as a function of the distance from the origin. When the velocity potential function is given we can get the velocity using this equation.